April 2013

ON A CLASS OF INTEGRABLE SYSTEMS WITH A QUARTIC FIRST INTEGRAL

Galliano VALENT * †

*Laboratoire de Physique Théorique et des Hautes Energies
Unité associée au CNRS UMR 7589
2 Place Jussieu, 75251 Paris Cedex 05, France

†Aix-Marseille Université et Université de Toulon
Centre de Physique Théorique
Unité associée au CNRS UMR 7332
Case 907, 13288 Marseille Cedex 9, France

Abstract

We generalize, to some extent, the results on integrable geodesic flows on two dimensional manifolds with a quartic first integral in the framework laid down by Selivanova and Hadeler. The local structure is first determined by a direct integration of the differential system which expresses the conservation of the quartic observable and is seen to involve a finite number of parameters. The global structure is studied in some details and leads to a class of models living on the manifolds $\mathbb{S}^2, \mathbb{H}^2$ or $\mathbb{R}^2$. As special cases we recover Kovalevskaya's integrable system and a generalization of it due to Goryachev.

1 Introduction

In 1999 Selivanova [6] has studied a class of integrable models, in two dimensional manifolds, with a quartic first integral which generalized Kovalevskaya's system. In a collaboration with Hadeler [2] the explicit local structure of these models was given and led to new globally defined systems on $\mathbb{S}^2$. The aim of this article is to study a generalization of these models and to determine which manifolds are involved.

The plan of the article is the following: in Section 2 we describe the general setting of our integrable models and solve the differential system giving their local structure. This requires to split the analysis in two cases, according to whether a parameter μ vanishes or not.

In Section 3 we analyze, in the case where μ vanishes, the global structure of the systems and the nature of the manifolds.

In Section 4 we consider the case where μ does not vanish. The integrable systems of Kovalevskaya and its generalization due to Goryachev appear in this class.

In Section 5 we prove that all of these models do not exhibit integrals of degree less or equal to three in the momenta.

In Section 6 some conclusions are presented, followed by Appendix A, devoted to a summary of definitions and formulas used throughout the article.

2 Local structure

Let us first present the general structure of the integrable systems to be dealt with.

2.1 The setting

We will start from the usual hamiltonian

$$H = K + V$$

where the kinetic energy and the potential are

$$K = \frac{1}{2}\Big(P_\theta^2 + a(\theta)\,P_\phi^2\Big), \qquad V = f(\theta)\,\cos\phi + g(\theta) \qquad\qquad f \not\equiv 0 \tag{1}$$

while the quartic integral will have the form $Q = Q_4 + Q_2 + Q_0$ where

$$\begin{aligned} &Q_4 = \lambda\,P_\phi^4 + 2\mu\,H\,P_\phi^2 && (\lambda,\,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\},\\ &Q_2 = \alpha\,\cos\phi\,P_\theta^2 + 2\beta(\theta)\,\sin\phi\,P_\theta\,P_\phi + (\gamma_0 + \gamma(\theta)\,\cos\phi)P_\phi^2 \\ &Q_0 = q(\theta,\phi). \end{aligned} \tag{2}$$

The parameters α and γ_0 are free and let us add the following comments:

1. The flow of H describes the geodetic motion for the metric

$$g = d\theta^2 + \frac{d\phi^2}{a(\theta)}. \tag{3}$$

We will consider only *riemannian* metrics.

2. The restriction $f \not\equiv 0$ is quite essential to obtain a truly quartic first integral. Indeed if f vanishes P_ϕ is conserved and Q_4 becomes reducible.

3. If one takes $\gamma_0(\theta)$ instead of a constant then its derivative must vanish for Q to be an integral. So we take γ_0 to be a constant.

4. If in Q_2 one takes $(\alpha_0(\theta)+\alpha(\theta)\,\cos\phi)\,P_\theta^2$, the two functions α_0 and α must be constants for Q to be an integral. Using the relation

$$\alpha_0\,P_\theta^2=\alpha_0\,(2H-a(\theta)\,P_\phi^2-V)$$

we see that the piece involving the hamiltonian is reducible while the term P_ϕ^2 can be included in γ_0 and the term V can be included in $q(\theta,\phi)$. Hence we have set $\alpha_0=0$.

5. If in Q_2 one adds a term $2\beta_0(\theta)\,P_\theta\,P_\phi$, with a non-vanishing β_0, then f must vanish for Q to be an integral. So we take $\beta_0(\theta)\equiv 0$.

6. Let us describe in this setting Kovalevskaya integrable system which is globally defined on $\mathbb{S}^2$. Using the $so(3)$ generators defined in Appendix A it is given by

$$H=\frac{1}{2}(L_1^2+L_2^2+2\,L_3^2)+k\,x\qquad Q=\left|\frac{1}{2}(L_1+iL_2)^2-k(x+iy)\right|^2, \tag{4}$$

easily transformed into

$$2H=P_\theta^2+\frac{1+\sin^2\theta}{\sin^2\theta}\,P_\phi^2+2k\,\sin\theta\,\cos\phi \tag{5}$$

which describes the geodesic flow on the metric

$$g=d\theta^2+\frac{\sin^2\theta}{1+\sin^2\theta}\,d\phi^2. \tag{6}$$

Its quartic first integral may be simplified into $\widehat{Q}\equiv Q-H^2$ giving

$$\widehat{Q}=P_\phi^4-2H\,P_\phi^2-2k\,\cos\theta\,\left(\sin\phi\,P_\theta+\frac{\cos\phi}{\tan\theta}\,P_\phi\right)P_\phi+k^2\,\sin^2\theta\,\sin^2\phi \tag{7}$$

which does fit with (2) for $\alpha=0$.

2.2 The differential system

It appears convenient to define, instead of θ, a new variable t such that $dt=a(\theta)d\theta$. The integrable system becomes

$$\left\{\begin{array}{l} H=\dfrac{1}{2}(a^2\,P_t^2+a\,P_\phi^2)+f\,\cos\phi+g\\[2mm] Q=\lambda\,P_\phi^4+2\mu\,H\,P_\phi^2+\left(\alpha a^2\,\cos\phi\,P_t^2+2\beta a\,\sin\phi\,P_t\,P_\phi+(\gamma_0+\gamma\,\cos\phi)P_\phi^2\right)+q.\end{array}\right. \tag{8}$$

Let us prove:

Theorem 1 *The constraint* $\{H,Q\}=0$ *is equivalent to the differential system*

$$\begin{array}{llll} (a) & \dot\beta=\dfrac{\alpha}{2}-\mu\,\dfrac{f}{a} & \dot\gamma=-2\beta+\alpha\,\dot a & \gamma+\beta\,\dot a=4\lambda\,\dfrac{f}{a}+2\mu\,f\\[2mm] (b) & \dot\Delta+2\alpha\,\dot f=-2\beta\,\dfrac{f}{a} & & \beta\,\dot g+\alpha\,g-(\gamma_0+2\mu\,g)\dfrac{f}{a}=L,\end{array} \tag{9}$$

where L *is some constant and*

$$\Delta=\beta\,\dot f-(\gamma+2\mu\,f)\frac{f}{a}.$$

Proof: In the Poisson bracket $\{H, Q\}$ the terms which would be of degree 5 in the momenta vanish identically. The terms of degree 3 give the equations (a) and the last piece of degree 1 yields

$$\partial_\phi\Big(\frac{q}{2}\Big) = \Big(\beta\,\dot{g} - (\gamma_0 + 2\mu\,g)\,\frac{f}{a}\Big)\sin\phi + \Delta\,\sin\phi\,\cos\phi$$

$$\partial_t\Big(\frac{q}{2}\Big) = \alpha\,\dot{f} + \alpha\,\dot{g}\,\cos\phi - \Big(\alpha\,\dot{f} + \beta\,\frac{f}{a}\Big)\sin^2\phi.$$

These partial differential equations give the two integrability conditions (b) in (9). When these relations hold we get

$$q = 2\alpha\,f + 2(\alpha\,g - L)\cos\phi + \Delta\,\sin^2\phi, \tag{10}$$

which conclude the proof. $\Box$

Let us now integrate this differential system, splitting the analysis in two cases.

2.3 First case: $\mu = 0$

Let us begin with:

Theorem 2 *The local structure of the integrable system is*

$$2\,H = \frac{1}{\beta^2}\Big(F\,P_x^2 + \sqrt{F}\,P_\phi^2 + \kappa\,\sqrt{F}\,G'\,\cos\phi + l\,G + m\,x + n\Big) \tag{11}$$

with

$$\beta^2 = b_0 + \alpha\,x \qquad F = x^4 + c_2\,x^2 + c_1\,x + c_0 \qquad G = \sqrt{F} - x^2 - \frac{c_2}{2}$$

and for the quartic integral

$$\begin{aligned} Q = P_\phi^4 + 2\kappa\Big(\alpha\frac{F}{\beta^2}\,\cos\phi\,P_x^2 + 2\,\sqrt{F}\,\sin\phi\,P_x\,P_\phi + (\alpha\frac{\sqrt{F}}{\beta^2} - 2x)\,\cos\phi\,P_\phi^2\Big) + 2l\,P_\phi^2 \\ +2\kappa^2\left(\alpha\,\frac{\sqrt{F}G'}{\beta^2}\,\cos^2\phi - G\,\sin^2\phi\right) + 2\kappa\Big(\alpha\,\frac{h}{\beta^2} - m\Big)\cos\phi. \end{aligned} \tag{12}$$

where

$$h = l\,G + m\,x + n$$

and all the constants are real.

Proof: The differential system (9) reduces to

$$\begin{aligned} &\dot{\beta} = \frac{\alpha}{2} \qquad \dot{\gamma} = -2\beta + \alpha\,\dot{a} \qquad \gamma + \beta\,\dot{a} = 4\lambda\,\frac{f}{a} \qquad \lambda \neq 0 \\ &D_t\left(\beta\,\dot{f} - \gamma\,\frac{f}{a} + 2\alpha\,f\right) = -2\beta\,\frac{f}{a} \qquad \beta\,\dot{g} + \alpha\,g = L + \gamma_0\,\frac{f}{a}. \end{aligned} \tag{13}$$

We will define a new variable $2x = \alpha\,a - \gamma$ for which we have $\dot{x} = \beta$ and we will demand that $\beta \not\equiv 0$. Denoting by a prime a derivative with respect to x we get from the first and the third relations in (13)

$$\beta^2 = b_0 + \alpha\,x \qquad 4\lambda\,\frac{f}{a} = (\beta^2\,a)' - 2x. \tag{14}$$

The fourth relation in (13), using the variable x, can be integrated once and gives

$$4\lambda\left(\beta^2\,f' + (\alpha\,a + 2x)\,\frac{f}{a}\right) = c_2 + 2x^2 - 2\beta^2\,a \tag{15}$$

where c_2 is a constant. Getting rid of f using the second relation in (14), gives an ODE for a which is

$$\left(\beta^4\,a^2\right)'' = 12x^2 + 2c_2 \qquad \Longrightarrow \qquad a = \frac{\sqrt{F}}{\beta^2} \qquad F = x^4 + c_2\,x^2 + c_1\,x + c_0, \tag{16}$$

where we took the positive root for a to secure the euclidean signature.

Upon the changes $\gamma_0 \to l$ and $L \to m/2$ we obtain

$$f = \frac{1}{4\lambda}\,\frac{\sqrt{F}\,G'}{\beta^2} \qquad g = \frac{1}{2\beta^2}\left(mx + n + \frac{l}{2\lambda}\,G\right) \qquad G = \sqrt{F} - x^2 - \frac{c_2}{2}, \tag{17}$$

and q follows from (10). Setting $\kappa = \dfrac{1}{2\lambda}$ and a few scalings give for the quartic integral the formula (12). $\square$

2.4 Second case: $\mu \neq 0$

In this case we have

Theorem 3 *The local form of the integrable system is*

$$2\,H = \frac{1}{\beta^2}\left(F\,P_x^2 + \sqrt{F}\,P_\phi^2 + \kappa\,\sqrt{F}\,G'\,\cos\phi + l\,G + mx + n\right) \tag{18}$$

with

$$\left\{\begin{array}{ll} F = (x^2+d)^2 - 4\,r\,p(x) & p(x) = \alpha\,x^3 + (c_2 - r\,\alpha^2)\,x^2 + c_1\,x + c_0 \\[2mm] \beta^2 = \dfrac{2\,p(x)}{x^2 + d + \sqrt{F}} & G = c_2 + \alpha\,x - \beta^2 \end{array}\right. \tag{19}$$

and for the quartic integral

$$\begin{aligned} Q &= r\,P_\phi^4 + 2\,H\,P_\phi^2 \\ &+\kappa\left(\alpha\frac{F}{\beta^2}\,\cos\phi\,P_x^2 + 2\,\sqrt{F}\,\sin\phi\,P_x\,P_\phi + (\alpha\frac{\sqrt{F}}{\beta^2} + 2r\,\alpha - 2x)\cos\phi\,P_\phi^2\right) + l\,P_\phi^2 \\ &+\kappa^2\left(\alpha\frac{\sqrt{F}\,G'}{\beta^2}\,\cos^2\phi - G\,\sin^2\phi\right) + \kappa\left(\alpha\frac{h}{\beta^2} - m\right)\cos\phi. \end{aligned} \tag{20}$$

where

$$h = l\,G + m\,x + n$$

and all the constants are real.

Proof: The differential system (9) becomes

$$\begin{array}{lll} \dot{\beta} = \dfrac{\alpha}{2} - \mu\,\dfrac{f}{a} & \dot{\gamma} = -2\beta + \alpha\,\dot{a}, & \gamma + \beta\,\dot{a} = 4\lambda\,\dfrac{f}{a} + 2\mu\,f \\[3mm] D_t\left(\beta\,\dot{f} - (\gamma + 2\mu f)\,\dfrac{f}{a} + 2\alpha\,f\right) = -2\beta\,\dfrac{f}{a} & & \beta\,\dot{g} + a\alpha\,g - (\gamma_0 + 2\mu g)\,\dfrac{f}{a} = L. \end{array} \tag{21}$$

where L is an integration constant.

Let us define a new variable

$$2x = \alpha\,(a + 2\,r) - \gamma \qquad \dot{x} = \beta \neq 0 \qquad r = \frac{\lambda}{\mu} \in \mathbb{R}. \tag{22}$$

Denoting by a prime a derivative with respect to the new variable x, the first and the third relations in (21) become

$$2\mu\,\frac{f}{a}=\alpha-(\beta^2)' \qquad\qquad 2\,r\,(\beta^2)'+(a\,\beta^2)'=2x \tag{23}$$

and the last one implies

$$a=\frac{x^2+d-2\,r\,\beta^2}{\beta^2}\equiv\frac{\sqrt{F}}{\beta^2} \qquad\qquad F=(x^2+d-2\,r\,\beta^2)^2 \tag{24}$$

where d is an integration constant.

The fourth relation in (21) can be written

$$D_t(\Delta+2\alpha\,f)=-2\beta\,\frac{f}{a} \qquad\qquad \Delta=\beta\,\dot{f}-(\gamma+2\mu\,f)\,\frac{f}{a}, \tag{25}$$

and when expressed in terms of the variable x it integrates up to

$$\beta^2\,(2\mu f)'+2\mu\,\frac{f}{a}\left(2x-2\,r\alpha+a\,(\beta^2)'\right)-2\beta^2+2\alpha x+2c_2=0 \tag{26}$$

with a new integration constant c_2. Using in this preceding relation the formulas (23) and (24), after some computations, one gets a simple ODE for β^2:

$$\Big(r\,(\beta^2)^2-(x^2+d)\,\beta^2\Big)''+6\alpha x+2(c_2-\rho\alpha^2)=0 \tag{27}$$

which is readily integrated to

$$r\,(\beta^2)^2-(x^2+d)\,\beta^2+p(x)=0 \qquad\qquad p(x)=\alpha\,x^3+(c_2-r\,\alpha^2)\,x^2+c_1\,x+c_0. \tag{28}$$

Inserting this result into (24) gives F and solving for β^2 we get the the relations given in (19).

One obtains for f and g

$$2f=\kappa\frac{\sqrt{F}G'}{\beta^2} \qquad\qquad 2g=\frac{h}{\beta^2} \qquad\qquad h=l\,G+mx+n$$

and one gets q using relation (10). After a few scalings one obtains the formula (20) for Q . □

Remarks:

1. The structure of this integrable model is therefore described by a finite number of parameters, playing different roles. Firstly we have the parameters which define the metric: α, b_0, c_0, c_1, c_2 in the first case and α, r, c_0, c_1, c_2, d in the second case; secondly we have the principal parameter κ which describes the bulk of the integrable system and thirdly we have secondary parameters l, m, n which creep in through g, as defined in formula (1), and are of minor interest.

2. Two integrable systems, with a quartic first integral, were derived in [5] and in [2]. The first one, with metric

$$g_1=\frac{dx^2}{a^2}+\frac{d\phi^2}{a} \qquad\qquad a=\sqrt{x^4+c_2\,x^2+c_1\,x+c_0} \tag{29}$$

corresponds to the special case $\alpha=l=m=n=0$ in Theorem 2. The second one had for metric

$$g_2=(a-x^2+p)\left(\frac{dx^2}{a^2}+\frac{d\phi^2}{a}\right), \tag{30}$$

where p is some constant. This metric corresponds to the special case $\alpha=l=m=n=0$ and $d=-p$ in our Theorem 3. It is difficult to push the comparison further since in these two references the existence of the quartic integral is proved but its explicit form is not given.

3. Tsiganov in [7] has given an integrable system with a quartic integral. It does not belong to our family since there is no $P_\theta \, P_\phi$ term in his quartic integral. This is forbidden for us since it would imply that the conformal factor β vanishes identically.

Let us turn ourselves to the study of the global structure and to the determination of the possible manifolds. In [2] the analysis was mostly interested in $M = \mathbb{S}^2$, however the non-compact manifolds $\mathbb{H}^2$ and $\mathbb{R}^2$ do appear.

The analysis (see [8]) is as follows: the positivity of F requires for x to be in some interval (a, b). We will intensively use the scalar curvature to establish whether the boundaries $x = a$ and $x = b$ are apparent coordinate singularities or true singularities forbidding a manifold. In the absence of true singularities the nature of the manifold is then determined by establishing a global conformal transformation between the actual metric and the canonical metrics, given in Appendix A, for $\mathbb{S}^2$ and $\mathbb{H}^2$.

Let us begin with the global analysis for the first case where μ vanishes.

3 The global structure for $\mu = 0$

This section will cover the integrable systems of Theorem 2, for which the metric is

$$g = \beta^2(x) \left(\frac{dx^2}{F} + \frac{d\phi^2}{\sqrt{F}} \right) \qquad\qquad \beta^2(x) = b_0 + \alpha\, x \tag{31}$$

and we will write

$$F(x) = x^4 + c_2\, x^2 + c_0 = (x^2 + a)(x^2 + \widetilde{a}) \qquad\qquad G = \sqrt{F} - x^2 - \frac{1}{2}(a + \widetilde{a}). \tag{32}$$

3.1 First case: $\alpha = 0$ and $c_1 = 0$

Since β^2 is constant we can set $b_0 = 1$. Let us first observe that the points $x = \pm\infty$ are apparent singularities of the metric (31). For instance $x = +\infty$ is mapped, by $u = 1/x$, to $u = 0+$ giving

$$g \sim du^2 + u^2\, d\phi^2$$

which is an apparent coordinate singularity due to the use of polar coordinates.
Let us begin the global analysis with :

Theorem 4 *The integrable system in Theorem 2:*

(i) Is trivial for $\widetilde{a} = a \in \mathbb{R}$.

(ii) Is not defined on a manifold if $\min(a, \widetilde{a}) < 0$ *and* $a \neq \widetilde{a}$.

(iii) Is defined on $\mathbb{H}^2$ *if* $\widetilde{a} = 0$ *and* $a > 0$. *It can be written* [1]

$$\left\{ \begin{array}{l} 2\,H = P_v^2 + \dfrac{c}{s^2}\, P_\phi^2 + \kappa\, \dfrac{s}{(c+1)^2}\, \cos\phi - l\, \dfrac{s^2}{(c+1)^2} \\[3mm] Q = P_\phi^4 - 4\kappa\Big(\sin\phi\, P_v + \cos\phi\, \dfrac{P_\phi}{s}\Big) P_\phi + 4l\, P_\phi^2 + \kappa^2\, \dfrac{s^2}{(c+1)^2}\, \sin^2\phi. \end{array} \right. \tag{33}$$

[1] In all that follows we will use the shorthand notation $s = \sinh\chi$, $c = \cosh\chi$ for hyperbolic functions.

(iv) Is defined on $\mathbb{S}^2$ if $0<\widetilde{a}<a$. It can be written [2]

$$\left\{\begin{array}{l} 2\,H=P_v^2+\dfrac{D}{S^2}\,P_{\phi}^2+\kappa\,k^4\,\dfrac{SC}{(D+1)^2}\,\cos\phi-l\,k^4\,\dfrac{S^2}{(D+1)^2}\qquad k^2\in\,(0,1)\\[3mm] Q=P_{\phi}^4-4\kappa\Big(\sin\phi\,P_v+\dfrac{C}{S}\,\cos\phi\,P_{\phi}\Big)P_{\phi}+4l\,P_{\phi}^2+\kappa^2\,k^4\,\dfrac{S^2}{(D+1)^2}\,\sin^2\phi. \end{array}\right. \tag{34}$$

(v) Is defined on $\mathbb{S}^2$ if $a\in\mathbb{C}\backslash\{0\}$ and $\widetilde{a}=\overline{a}$. It can be written

$$\left\{\begin{array}{l} 2\,H=P_v^2+\dfrac{\mu}{S^2}\,P_{\phi}^2-\kappa\,k^2\,k'^2\,\dfrac{SC}{D^3}\,\cos\phi+l\,k^2\,k'^2\,\dfrac{S^2}{D^2}\qquad k^2\in\,(0,1)\\[3mm] Q=P_{\phi}^4-\kappa\left(\sin\phi\,P_v+\dfrac{C}{SD}\,\cos\phi\,P_{\phi}\right)P_{\phi}+l\,P_{\phi}^2-\kappa^2\,k^2\,k'^2\,\dfrac{S^2}{4\,D^2}\,\sin^2\phi \end{array}\right. \tag{35}$$

with

$$\mu=1-k^2\,\frac{S^2\,C^2}{D^2}\geq k'^2.$$

Proof of (i):
In this case we have $G\equiv 0$ which implies, as observed in Section 2.1, that P_{ϕ} is conserved and so that Q_4 is reducible. Using the coordinate $u=1/x$ the metric

$$g=\frac{du^2}{(1+a\,u^2)^2}+\frac{u^2\,d\phi^2}{1+a\,u^2}\qquad\qquad a\in\mathbb{R},$$

is of constant scalar curvature since $R=2a$. The discussion is then

1. For $a>0$ the change $\sqrt{a}\,u=\tan\theta$ gives the canonical metric (105) on $\mathbb{S}^2$.
2. For $a=0$ we get $\mathbb{R}^2$ with its flat metric.
3. For $a<0$ the change $\sqrt{|a|}\,u=\tanh\chi$ gives the canonical metric (106) on $\mathbb{H}^2$.

One can check that Q itself is always fully reducible, trivializing the integrable system. □

Proof of (ii):
Taking $a<\widetilde{a}$ we have to consider two cases:

$$(1):\quad a=-x_2^2,\quad \widetilde{a}=-x_1^2\qquad\qquad (2):\quad a=-x_2^2,\quad \widetilde{a}=x_1^2\quad\text{with}\quad 0<x_1<x_2.$$

In both cases positivity allows $x\in(x_2,+\infty)$ and we need to study the nature of the singularity at $x=x_2$. It can be ascertained from the scalar curvature

$$R=-\frac{3}{4}\,x_2(x_2^2+\widetilde{a})\,\frac{1}{x-x_2}+O(1)$$

which shows that it is a true singularity of the metric, forbidding any manifold.

In the first case positivity also allows $x\in(-x_1,+x_1)$. In this case we have

$$R=-\frac{3}{4}\,x_2(x_2^2-x_1^2)\,\frac{1}{x-x_1}+O(1)$$

leading to the same conclusion. □

[2] In all that follows we will use the shorthand notation $S=\mathrm{sn}\,(\theta,k^2),\ \ C=\mathrm{cn}\,(\theta,k^2),\ \ D=\mathrm{dn}\,(\theta,k^2)$ for Jacobi's elliptic functions.

Proof of (iii):
Here we have $a \in \mathbb{R}\backslash\{0\}$. Up to a scaling of the observables, we can set $|a| = 1$. Using again the coordinate $u = 1/x$ the metric becomes

$$g = \frac{du^2}{1+\epsilon\, u^2} + \frac{u^2}{\sqrt{1+\epsilon\, u^2}}\, d\phi^2 \qquad\qquad \epsilon = \text{sign}\,(a). \tag{36}$$

For $\epsilon = -1$ we have $u \in (0,1)$. However for $u \to 1-$ the scalar curvature

$$R = -\frac{3}{4(1-u)} + O(1)$$

exhibits a true singularity precluding any manifold.

For $\epsilon = +1$ we have $u \in (0,+\infty)$. As already observed $u = 0+$ is an apparent singularity. The change of variable $u = \sinh v$ gives for metric and scalar curvature

$$g = dv^2 + \frac{\sinh^2 v}{\cosh v}\, d\phi^2 \qquad\qquad R = 1 - \frac{3}{2}\, \tanh^2 v \qquad v \in (0,+\infty) \quad \phi \text{ azimuthal} \tag{37}$$

showing that the manifold is $M = \mathbb{H}^2$.

In the hamiltonian we can set $n = 0$ but we must take $m = 0$ for the potential to be defined on M. Transforming H and Q into the coordinates $(v,\, \phi,\, P_v,\, P_\phi)$ gives (33).

Let us prove that this integrable system is globally defined on $M = \mathbb{H}^2$. Writing the metric (37) as

$$g = \frac{\sinh^2 v}{\cosh v}\left(d\phi^2 + \frac{\cosh v}{\sinh^2 v}\, dv^2\right)$$

if we define a new coordinate χ by

$$\frac{d\chi}{\sinh\chi} = \frac{\sqrt{\cosh v}}{\sinh v}\, dv \qquad\qquad v \in\ (0,+\infty)\ \ \to\ \ \chi \in\ (0,+\infty) \tag{38}$$

we get, using the formulas of Appendix A:

$$g = \Omega^2\, g(H^2, \text{can}) \qquad\qquad \Omega^2 = \frac{(1-t^2)^2\, \sinh^2 v}{4t^2\, \cosh v} \qquad\quad t \equiv \tanh\frac{\chi}{2}\ \in\ (0,1) \tag{39}$$

where

$$t = \tanh\frac{v}{2}\; e^{\eta(v)} \qquad\qquad \eta(v) = -\int_v^{+\infty} \frac{(\sqrt{\cosh x} - 1)}{\sinh x}\, dx. \tag{40}$$

It follows that the first relation in (40) can be extended to $v \in \mathbb{R}$. Then the function $t(v)$ is odd (while $\eta(v)$ is even), C^∞ and strictly increasing. This implies that its reciprocal function $v(t)$ is continuous, odd and strictly increasing for $t \in (-1,1)$. Since

$$D_v\, t = \frac{\sqrt{\cosh v}}{1+\cosh v}\, e^{\eta(v)}$$

never vanishes $v(t)$ is also C^∞ for $t \in (-1,+1)$. It follows that $\eta \circ v(t)$ is an even continuous function of t so we can define $\eta\circ v(t) = \psi(t^2)$. The function ψ is C^∞ for $t^2 \in (0,1)$ but we need to extend it to $[0,1)$. Since for $t \to 0+$ we have $v(t) \to 0+$, an easy expansion in powers of $v(t)$ shows that

$$\psi' \equiv D_{t^2}\psi = \frac{\eta'(v(t))}{D_v\,(t^2)} = c_0^{(1)} + c_1^{(1)}\, v(t)^2 + O(v(t)^4).$$

This structure may be shown to hold for all the derivatives of ψ with respect to the variable t^2 by recurrence. As a consequence we have

$$\tanh\frac{v}{2} = t\,e^{-\psi(t^2)} \qquad\qquad t\in[0,1) \tag{41}$$

and any C^∞ function $f(v^2)$ for $v\in[0,+\infty)$ can be written $\widetilde{f}(t^2)$ which will be a C^∞ function of $t^2\in[0,1)$. Recalling that

$$t^2 = \frac{\eta_3-1}{\eta_3+1}$$

the function $f(v^2)$ belongs to $C^\infty(M)$. Close to $t=0$ the Taylor expansion

$$v = \tau - \frac{\tau^3}{24} + O(\tau^5) \qquad\qquad \tau = 2e^{-\eta(0)}\,t \tag{42}$$

will be useful.

We are now in position to prove that the integrable system given by (33) is globally defined. Using the generators M_i $i=1,2,3$ in T^*_M of the isometries one can write the hamiltonian

$$2H = \frac{1}{\Omega^2}\Big(M_1^2+M_2^2-M_3^2\Big) + \kappa\,\Lambda_1\,\eta_1 - l\,\Lambda_2. \tag{43}$$

From the previous argument all the functions in H belong to $C^\infty(M)$ as can be seen from their formulas

$$\Lambda_1(t^2) = \frac{(1-t^2)\,\sinh(v(t))}{2t\,(\cosh(v(t))+1)^2} \qquad\qquad \Lambda_2(t^2) = \frac{\sinh^2(v(t))}{(\cosh(v(t))+1)^2}$$

and for $t\to 0+$ the check just uses relation (42). The cubic observable can be written

$$Q = M_3^4 - 4\kappa\Big(\Lambda_3(t^2)\,M_1 - \Lambda_4(t^2)\,\eta_1\,M_3\Big)M_3 + \kappa^2\,\Lambda_5(t^2)\,(\eta_2)^2 + 4l\,M_3^2. \tag{44}$$

and by the same argument all the functions $\Lambda_i(t^2)$ do belong to $C^\infty(M)$. $\square$

Proof of (iv):
The function $F=(x^2+a)(x^2+\widetilde{a})$ is strictly positive and the metric is

$$g = \frac{dx^2}{F} + \frac{d\phi^2}{\sqrt{F}} \qquad\qquad x\in\mathbb{R} \qquad \phi\ \text{ azimuthal} \tag{45}$$

with a scalar curvature

$$R = a+\widetilde{a} - \frac{3}{2}\,\frac{\delta\,x^2}{F} \qquad\qquad \delta=(a-\widetilde{a})^2$$

which is C^∞ for all $x\in\mathbb{R}$. As already explained at the beginning of this section the points $x=\pm\infty$ are apparent singularities, geometrically the poles of a sphere. Since the x coordinate gives untransparent expressions for F, G and G' we will use Jacobi elliptic functions. Let us consider the metric written in the variable $u=x^2\in(0,+\infty)$. We get first

$$g = \frac{du^2}{4u(u+a)(u+\widetilde{a})} + \frac{d\phi^2}{\sqrt{(u+a)(u+\widetilde{a})}}.$$

The change of variables

$$u = a\,\frac{\mathrm{cn}^2(v,k^2)}{\mathrm{sn}^2(v,k^2)} \qquad\qquad \widetilde{a}=(1-k^2)\,a \qquad k^2\in(0,1)$$

gives for the metric (no loss by taking $a=1$):

$$g = dv^2 + \frac{S^2}{D}\, d\phi^2 \qquad\qquad v \in (0,K) \qquad \phi \ \text{ azimuthal.} \tag{46}$$

using our earlier shorthand notations for Jacobi elliptic functions. However when going to the variable u we have lost half of the manifold and this is why $v \in (0,K)$. But now we can recover the full manifold by extending $v \in (0,2K)$ since $v=0$ and $v=2K$ are indeed apparent singularities (the "poles" of the manifold). The scalar curvature

$$2R = 3\,D^2 - 1 - k'^2 + 3\,\frac{k'^2}{D^2}$$

is indeed $C^\infty([0,2K])$. In the hamiltonian we can set $n=0$ and we must impose $m=0$ since this piece is singular for $u \to 0+$. Then transforming (H,Q) into the coordinates $(v,\,\phi,\,P_v,\,P_\phi)$ gives the formulas (34).

This integrable system is globally defined on $M=\mathbb{S}^2$; to prove this let us define

$$t \equiv \tan\frac{\theta}{2} = \frac{S(v)}{C(v)+D(v)}\, e^{\eta(v)} \qquad\qquad \eta(v) = \int_K^v \frac{\sqrt{D(x)}-1}{S(x)}\, dx \tag{47}$$

which maps $v \in (0,2K) \to t \in (0,+\infty)$. As already explained in the proof of (iii) we have

$$g = \Omega^2\ g(S^2,\text{can}) \qquad\qquad \Omega^2(t^2) = \frac{(1+t^2)^2\, S^2(v(t))}{4t^2\, D(v(t))}. \tag{48}$$

The first relation in (47) can be extended to $v \in (-2K,2K)$ or $t \in \mathbb{R}$. Then $t(v)$ is odd, C^∞ and strictly increasing. Hence $v(t)$ is odd, continuous and strictly increasing. Since

$$t' = D_v\, t = \frac{1+\sqrt{D(v)}}{C(v)+D(v)}\, e^{\eta(v)}$$

never vanishes $v(t)$ is in fact C^∞ and $\eta\circ v(t)$ is an even continuous function of t so we can define $\eta\circ v(t) = \psi(t^2)$. By the same argument detailed in the proof of (iii) the function ψ is C^∞ for $t^2 \in [0,+\infty)$ and so any C^∞ function $f(v^2)$ for $v \in [0,2K)$ can be written $\widetilde{f}(t^2)$. Since we have

$$t^2 = \frac{1-\zeta_3}{1+\zeta_3}$$

it follows that $\widetilde{f}(t^2)$ will be a C^∞ function on the manifold, except at the south pole. This is not a problem since the relation

$$t(2K-v) = \frac{1}{t(v)} \qquad \Longrightarrow \qquad v(1/t) = 2K - v(t) \tag{49}$$

relates the behaviour at the north pole ($v=0;\ t=0;\ \zeta_3=+1$) with the behaviour at the south pole ($v=2K;\ t=+\infty;\ \zeta_3=-1$). So it will be sufficient to consider $t \in [0,+\infty)$ and analyze what happens at the north pole using the Taylor expansion

$$v(t) = \tau - \frac{(1+k'^2)}{24}\,\tau^3 + O(\tau^5) \qquad\qquad \tau = e^{-\eta(0)}\, t. \tag{50}$$

Let us first examine the hamiltonian

$$2H = \frac{1}{\Omega^2(t^2)}\Big(L_1^2+L_2^2+L_3^2\Big) + \kappa\, k^4\, \Lambda_1(t^2)\,\zeta_1 - l\, k^4\, \Lambda_2(t^2) \tag{51}$$

with

$$\Lambda_1(t^2) = \frac{(1+t^2)\,S(v(t))\,C(v(t))}{2t\,(D(v(t))+1)^2} \qquad\qquad \Lambda_2(t^2) = \frac{S^2(v(t))}{(D(v(t))+1)^2}.$$

These three functions are C^∞ for $t \in (0,+\infty)$ and upon use of (50) for $t \in [0,+\infty)$ they will belong to $C^\infty(M)$. For the quartic integral

$$Q = L_3^4 - \kappa\Big(\Lambda_3(t^2)\,L_1 + \Lambda_4(t^2)\,\zeta_1\,L_3\Big)L_3 + \kappa^2\,k^4\,\Lambda_5(t^2)\,(\zeta_2)^2. \tag{52}$$

the argument is similar. $\square$

Proof of (v):
The first change of coordinate $u = x^{-2}$ gives for metric

$$g = \frac{du^2}{4u\,p(u)} + \frac{u}{\sqrt{p(u)}}\,d\phi^2 \qquad\qquad p(u) = (1+a\,u)(1+\overline{a}\,u) \qquad u \in (0,+\infty) \tag{53}$$

while the second change

$$\mathrm{cn}\,(2v) = \frac{1-|a|\,u}{1+|a|\,u} \qquad\qquad k^2 = \frac{1}{2}\left(1 - \frac{a+\overline{a}}{2\,|a|}\right) \qquad v \in (0,K) \tag{54}$$

transforms it into

$$g = \frac{1}{|a|}\left(dv^2 + \frac{S^2}{\mu}d\phi^2\right) \qquad\qquad \mu = 1 - k^2\frac{S^2\,C^2}{D^2} \geq \frac{2k'}{1+k'}. \tag{55}$$

By a global scaling we can set $|a| = 1$. The transition from $x \to u$ again loses half of the manifold. To recover it we will extend $v \in (0,2K)$ and ϕ azimuthal since both end-points $v = 0$ and $v = 2K$ are just the coordinate singularities corresponding to the "poles"' of $\mathbb{S}^2$. The scalar curvature

$$\frac{R}{2} = k'^2 - k^2 + \frac{12k'^4}{\mu^2}\,\frac{(2D^2-1)}{D^4} - \frac{12k'^2}{\mu}\,\frac{D^2-k^2}{D^2}$$

is again $C^\infty([0,2K])$.

The computations of G and $\sqrt{F}\,G'$ are tricky but give eventually the simple results

$$\sqrt{F}\,G' = -4\,k^2\,k'^2\,\frac{S\,C}{D^3} \qquad\qquad G = 2\,k^2\,k'^2\,\frac{S^2}{D^2}.$$

Again we have to set $m = n = 0$ and the coordinates change $(x,\,\phi,\,P_x,\,P_\phi) \to (v,\,\phi,\,P_v,\,P_\phi)$ gives for H and Q the formulas (35) up to a scaling of κ.

To prove the global definiteness on $M = \mathbb{S}^2$ let us give the key formulas needed. Defining

$$t \equiv \tan\frac{\theta}{2} = \frac{k'\,S(v)}{C(v)+D(v)}\,e^{\eta(v)} \qquad \eta(v) = \int_K^v \frac{\sqrt{\sigma(x)}-1}{S(x)}\,dx \tag{56}$$

we have

$$g = \Omega^2\,g(S^2,\mathrm{can}) \qquad\qquad \Omega^2(t^2) = \frac{(1+t^2)^2\,S^2(v(t))}{4t^2\,\sigma(v(t))}. \tag{57}$$

The relations (49) are still valid and the check of C^∞-ness at the north pole needs

$$v(t) = \tau + \frac{(5k^2-1)}{12}\,\tau^3 + O(\tau^5) \qquad\qquad \tau = e^{-\eta(0)}\,t.$$

Writing the hamiltonian as

$$2H = \frac{1}{\Omega^2(t^2)}\left(L_1^2 + L_2^2 + L_3^2\right) - \kappa\,k^2\,k'^2\,\Lambda_1(t^2)\,\zeta_1 + \kappa\,k^2\,k'^2\,\Lambda_2(t^2) \tag{58}$$

and the quartic integral

$$Q = L_3^4 - \kappa\Big(\Lambda_3(t^2)\, L_1 + \Lambda_4(t^2)\,\zeta_1\, L_3\Big) L_3 - \kappa\, k^2\, k'^2\, \Lambda_5(t^2)\,(\zeta_2)^2 \tag{59}$$

one can check that all the $\Lambda_i(t^2)$ are C^∞ functions for $t^2 \in [0, +\infty[$ implying that the integrable system is globally defined on $\mathbb{S}^2$. $\square$

Remarks:

1. The appearance of a constant curvature metric in case (i) was already observed in [2].

2. In [2] the case $F = x^4 + a\,x^2 + 1$ is discussed in their Theorem 1. However their restriction $a > -2$ for the hamiltonian to be globally defined on $M = \mathbb{S}^2$ should be modified to $-2 < a < 2$.

3. If in the integrable system (34) we take the limit $k^2 \to 1$ we smoothly recover the system (33). However the manifold changes drastically.

4. The cases (iv) and (v) can be fully analyzed in one stroke if one keeps the coordinate x. However the structure of the observables appears to be simpler using elliptic functions.

3.2 Second case: $\alpha \neq 0$ and $c_1 = 0$

Setting $\alpha = 1$ let us write the conformal factor $\beta^2(x) = x - x_0$. The metric is

$$g = (x - x_0)\left(\frac{dx^2}{F} + \frac{d\phi^2}{\sqrt{F}}\right) \qquad\qquad F(x) = (x^2 + a)(x^2 + \widetilde{a}). \tag{60}$$

Let us first examine what is changing for $x \to +\infty$. The metric becomes

$$g \sim \frac{dx^2}{x^3} + \frac{d\phi^2}{x} = 4(d\tau^2 + \tau^2\, d\Phi^2) \qquad \tau = \frac{1}{\sqrt{x}} \to 0+ \qquad \Phi = \frac{\phi}{2}$$

and if we take Φ to be azimuthal we see that $x = +\infty$ is again an apparent singularity. This gives $\cos(2\Phi)$ for the angular dependence of the potential.

Theorem 5 *The integrable system in Theorem 2:*

(i) Is trivial for $\widetilde{a} = a$.

(ii) Is not defined on a manifold if $\min(a, \widetilde{a}) < 0$ and $a \neq \widetilde{a}$.

(iii) Is defined on $\mathbb{H}^2$ if $\widetilde{a} = 0$ and $a > 0$. The hamiltonian becomes

$$2H = \frac{1}{B}\left(A^2\, P_v^2 + A\,\frac{P_\Phi^2}{v^2} + \kappa\,\frac{v^4}{(A+1)^2}\,\cos(2\Phi) - l\,\frac{v^6}{(A+1)^2} + m + n\,v^2\right) \tag{61}$$

with

$$A = \sqrt{1 + v^4} \qquad\qquad B = 1 + \rho\, v^2$$

and the quartic observable

$$\begin{aligned} Q = P_\phi^4 - \frac{4\kappa}{B}\,\mathcal{M}^2 + \frac{4\kappa}{B}\,(B^2 - 2B + A)\,\cos^2\Phi\,\frac{P_\Phi^2}{v^2} + \frac{4\kappa}{B}\,(B - A)\,\frac{P_\Phi^2}{v^2} + 4l\,P_\Phi^2 \\ -\frac{\kappa^2}{B}\frac{(2-B)}{(A+1)^2}\,v^4\,\sin^2(2\Phi) + 4\frac{\kappa}{B}\Big(\frac{v^2}{(A+1)^2}(\kappa + l\,v^2) + \rho\, m - n\Big)\,v^2\,\sin^2\Phi \end{aligned} \tag{62}$$

with

$$\mathcal{M} = A\,\sin\Phi\, P_v + B\,\cos\Phi\,\frac{P_\Phi}{v}.$$

(iv) Is not defined on a manifold for $0 < a < \widetilde{a}$ *or* $\widetilde{a} = \overline{a}$*.*

Proof of (i):
Since $G = \sqrt{F} - x^2 - (a+\widetilde{a})/2$ vanishes identically P_ϕ is conserved so that Q_4 becomes reducible, trivializing the system.
Proof of (ii):
Taking $a < \widetilde{a}$ we have to consider two cases:

$$(1): \quad a = -x_2^2, \quad \widetilde{a} = -x_1^2 \qquad\qquad (2): \quad a = -x_2^2, \quad \widetilde{a} = x_1^2 \quad \text{with} \quad 0 < x_1 < x_2.$$

In both cases positivity is ensured if we take $x \in (x_2, +\infty)$. Let us consider again the metric behaviour at the end-point $x = x_2+$. If the conformal factor does not vanish for $x \to x_2+$ then it does not change the nature of the singularity, which remains a true one as already proved in Theorem 4 (ii). If the conformal factor vanishes as $x - x_2$ then the scalar curvature

$$R = -\frac{3x_2}{4}(x_2^2 + \widetilde{a})\,\frac{1}{(x-x_2)^2} + O\left(\frac{1}{x-x_2}\right)$$

shows that x_2 remains a true singularity.

In the first case positivity may be also satisfied with $x \in (-x_1, +x_1)$. If the conformal factor does not vanish for $x \to x_1-$ then it does not change the nature of the singularity, which remains a true one. If the conformal factor vanishes as $x - x_1$ then the scalar curvature

$$R = -\frac{3x_2}{4}(x_2^2 - x_1^2)\,\frac{1}{(x-x_1)^2} + O\left(\frac{1}{x-x_1}\right)$$

shows that x_1 remains singular. □
Proof of (iii):
In the metric (60) we can set $|a| = 1$. The change of variable $x = 1/v^2$, up to a global scaling, brings it into the form

$$g = (1 - x_0\, v^2)\left(\frac{dv^2}{1+\epsilon\, v^4} + \frac{v^2}{\sqrt{1+\epsilon\, v^4}}\, d\Phi^2\right) \qquad \epsilon = \text{sign}\,(a) \qquad \Phi = \frac{\phi}{2} \tag{63}$$

Let us first consider the case $\epsilon = -1$ which requires $v \in (0,1)$. The curvature is singular for $v \to 1-$ since we have

$$R = -\frac{3}{2(1-x_0)}\,\frac{1}{1-v} + O(1)$$

and for $x_0 = 1$ the singularity is even worse

$$R = \frac{3}{4}\,\frac{1}{(1-v)^2} + O\left(\frac{1}{1-v}\right)$$

so there can be no manifold.

Let us consider now the case $\epsilon = +1$ for which $v \in (0, +\infty)$ and Φ is an azimuthal angle. If $x_0 > 0$ let us define $x_0 = 1/\nu^2$; then positivity of the conformal factor requires $v \in (0, \nu)$ and the curvature is singular for $v \to \nu-$ since

$$R = -\frac{\nu(\nu^4+1)}{8}\,\frac{1}{(v-\nu)^3} + O\left(\frac{1}{(v-\nu)^2}\right)$$

For $x_0 \leq 0$ let us define $x_0 = -\rho$. Now the conformal factor never vanishes and the metric becomes

$$g = (1 + \rho\, v^2)\left(\frac{dv^2}{1+v^4} + \frac{v^2}{\sqrt{1+v^4}} d\Phi^2\right) \qquad v \in (0, +\infty) \qquad \Phi \quad \text{azimuthal} \tag{64}$$

and $v=0$ is an apparent singularity. The scalar curvature

$$R=-\frac{(\rho\,v^6-1)(\rho\,v^4+3v^2-2\rho)}{2(v^4+1)(1+\rho\,v^2)^2}$$

is now $C^{\infty}([0,+\infty))$.

Using for coordinates $(v,\,P_v,\,\Phi,P_\Phi)$ in T_M^* a lengthy computation leads to the integrable system given by (61) and (62), up to scalings and the elimination of a reducible piece proportional to the hamiltonian.

Let us now prove that the manifold is $M=\mathbb{H}^2$ and that the system is globally defined on M. As already explained in the proof of Theorem 4 case (iii), if we define a new coordinate t by

$$t\equiv\tanh\frac{\chi}{2}=\frac{v}{1+\sqrt{1+v^2}}\,e^{\eta(v)}\qquad \eta(v)=-\int_v^{+\infty}\left(\frac{1}{\sqrt[4]{1+x^4}}-\frac{1}{\sqrt{1+x^2}}\right)\frac{dx}{x}\tag{65}$$

mapping $v\in\,(0,+\infty)$ into $t\in\,[0,1)$ we have

$$g=\Omega^2\,g(H^2,\mathrm{can})\qquad\qquad \Omega^2=\frac{v^2}{4t^2}\frac{1+\rho\,v^2}{\sqrt{1+v^4}}(1-t^2)^2.\tag{66}$$

The proof that $\eta\circ v(t)=\psi(t^2)$ is C^∞ for $t\in[0,1)$ is fully similar to the one given in some detail for the proof of Theorem 4 case (iii). Writing the hamiltonian

$$2H=\frac{1}{\Omega^2}\Big(M_1^2+M_2^2-M_3^2\Big)+2\kappa\,\Lambda_1\,(\eta_1)^2-(\kappa+l\,v^2)\,C+m+n\,v^2\tag{67}$$

all functions of v^2 belong to $C^\infty(M)$ as well as

$$\Lambda_1=C\,\frac{(1-t^2)^2}{4t^2}\qquad\qquad C=\frac{v^4}{(A+1)^2}\qquad\qquad A=\sqrt{1+v^4}.$$

The quartic first integral can be written

$$Q=M_3^4+\frac{4\kappa}{B}\,\mathcal{M}^2-\frac{4\kappa}{B}\,\Lambda_2\,(\eta_1)^2\,M_3^2+(\Lambda_3+4l)M_3^2-\frac{4\kappa^2}{b}\,\Lambda_4\,(\eta_1\,\eta_2)^2+\frac{4\kappa}{B}\,\Lambda_5\,(\eta_2)^2$$

with

$$\Lambda_2=\frac{(1-t^2)^2(B^2-2B+A)}{4v^2\,t^2}\qquad \Lambda_3=\frac{B-A}{v^2}\qquad \Lambda_5=\frac{v^2(1-t^2)^2}{4t^2}\left((k+l\,v^2)\frac{C}{v^2}+\rho\,m-n\right)$$

and

$$\Lambda_4=\frac{(1-t^2)^2(2-B)C}{4t^2}\qquad\quad \mathcal{M}=\frac{2t}{v}\frac{\sqrt{A}}{1-t^2}\left\{M_1+\frac{(1-t^2)^2}{4t^2}\left(\frac{B}{\sqrt{A}}-\frac{1+t^2}{1-t^2}\right)\,\eta_1\,M_3\right\}$$

Taking into account that for $v\to 0+$ we have

$$B-A\sim\rho\,v^2\qquad\qquad B^2-2B+A\sim(\rho^2+1/2)v^4\qquad\qquad C\sim v^4/4$$

we conclude that $\mathcal{M}$ is globally defined and that all the functions Λ_i do belong to $C^\infty(M)$. $\square$

Proof of (iv):

No matter how we reduce the interval for x to ensure the positivity of the conformal factor it will begin or end at the zero of $x-x_0$. The vanishing of the conformal factor induces a true singularity as witnessed by the scalar curvature

$$R=(x_0^2+a)(x_0^2+\widetilde{a})\,\frac{1}{(x-x_0)^3}+O\left(\frac{1}{(x-x_0)^2}\right)$$

so there is no manifold for this metric. $\square$

3.3 Third case: $c_1 \neq 0$

In this case we have:

Theorem 6 *For $c_1 \neq 0$, no matter what α is, the integrable models given by Theorem 2 become globally trivial.*

Proof:
The proofs of Theorems 4 and 5 have shown that $x = +\infty$ is always an apparent singularity which is either the "pole" of $\mathbb{H}^2$ or one of the "poles" of $\mathbb{S}^2$, when a manifold is available. Let us consider the second case, the analysis of the first case being fully similar. The coordinate t on the sphere is given by

$$t \equiv \tan\frac{\theta}{2} = \exp\left(\eta_0 - \int \frac{dx}{\sqrt[4]{F}}\right)$$

and $x \to +\infty$ is mapped into $t \to 0+$. One gets

$$\tau \equiv e^{-\eta_0}\, t = \frac{1}{x} + O\left(\frac{1}{x^3}\right) \qquad \Longrightarrow \qquad \frac{1}{x} = e^{-\eta_0}\, t + O(t^3)$$

Let us consider in Q the term

$$\kappa^2\, G\, \sin^2\phi = 4\kappa^2\, G\, \sin^2\Phi\, \cos^2\Phi = \kappa^2\, \frac{(1+t^2)^2\, G}{t^2}\, (\zeta_1\, \zeta_3)^2.$$

The relations

$$G = \frac{c_1}{2x} + O\left(\frac{1}{x^2}\right) \qquad \Longrightarrow \qquad \frac{G}{t^2} = \frac{c_1\, e^{-\eta_0}}{2t} + O(1)$$

show that this term is not even defined for $t = 0$ so we must impose $\kappa = 0$. Then P_ϕ becomes conserved and Q_4 is trivialized.

Up to now we have just proved that any interval which includes $\pm\infty$ will not give a globally defined integrable system. However there remains the possibility that the manifold be recovered for some bounded interval (x_2, x_1) where F vanishes at both end-points.

So let us first consider the case where x_1 is a simple zero and $\alpha = 0$ hence $\beta^2 = 1$. We can write $F = (x_1 - x)P(x)$ where P is a cubic polynomial such that $P(x_1) \neq 0$. The curvature

$$R = -\frac{3}{8}\, \frac{P(x_1)}{x_1 - x} + O(1)$$

is singular for $x \to x_1-$.

If $\alpha \neq 0$ either $\beta^2 = x_0 - x$ does not vanish for $x = x_1$ and the curvature remains singular or $x_0 = x_1$ and then the singularity is even worse

$$R = \frac{3}{8}\, \frac{P(x_1)}{(x_1 - x)^2} + O\left(\frac{1}{x_1 - x}\right)$$

If x_1 is threefold there cannot be a bounded interval of positivity for F and we are left with the single possibility that the zeroes x_1 and x_2 be twofold with $F = (x - x_1)^2 (x - x_2)^2$. Since the third power of x is absent from F we get $x_2 = -x_1$ leading to $F = (x^2 - x_1^2)^2$ for which $c_1 = 0$, a contradiction. $\square$

4 The global structure: $\mu \neq 0$

We will now consider the integrable systems of Theorem 3. Let us begin with the simpler case where $r = 0$.

4.1 First case: $r=0$

The metric becomes

$$g=\frac{p(x)}{(x^2+d)^2}\left(\frac{dx^2}{x^2+d}+d\phi^2\right)\qquad\qquad p(x)=\alpha\,x^3+c_2\,x^2+c_1\,x+c_0 \tag{68}$$

By a scaling we can set $d=\epsilon$ with either $\epsilon=0$ or $\epsilon=\pm1$. Switching to $u=1/x$ we get

$$g=\frac{u^2\,p(1/u)}{1+\epsilon\,u^2}\left(\frac{du^2}{(1+\epsilon\,u^2)^2}+\frac{u^2\,d\phi^2}{1+\epsilon\,u^2}\right). \tag{69}$$

It follows that we have:

Theorem 7 *The integrable system described by Theorem 3:*

(i) For $\epsilon=0$ is defined on $M=\mathbb{R}^2$ and becomes

$$\left\{\begin{array}{l} 2H=\dfrac{1}{B}\Big(P_x^2+P_y^2+2\,\kappa\,x+l\,(x^2+y^2)+m\Big)\\ Q=2\rho\,H\,L_3^2-2\kappa\,P_y\,L_3-l\,L_3^2+\kappa^2\,y^2\qquad\qquad L_3=x\,P_y-y\,P_x\end{array}\right. \tag{70}$$

with

$$B=1+\rho(x^2+y^2)\qquad\qquad \rho>0$$

(ii) For $\epsilon=-1$ is defined on $M=\mathbb{H}^2$ and becomes

$$\left\{\begin{array}{l} 2H=\dfrac{1}{B}\Big(M_1^2+M_2^2-M_3^2+2\kappa\,\eta_1\,\eta_3+l\,\eta_3^2+m\Big)\\ Q=4\rho\,H\,M_3^2-2\kappa\,M_1\,M_3-l\,M_3^2+\kappa^2\,\eta_2^2\end{array}\right. \tag{71}$$

with

$$B=1-\rho+2\rho\,\eta_3^2\qquad\qquad \rho\in\ (-1,0)\cup(0,+\infty).$$

(iii) For $\epsilon=+1$ is defined on $M=\mathbb{S}^2$ and becomes

$$\left\{\begin{array}{l} 2H=\dfrac{1}{B}\Big(L_1^2+L_2^2+L_3^2+2\kappa\,\zeta_1\,\zeta_3+l\,\zeta_3^2+m\Big)\\ Q=4\rho\,H\,L_3^2+2\kappa\,L_1\,L_3-l\,L_3^2-\kappa^2\,\zeta_2^2\end{array}\right. \tag{72}$$

with

$$B=1-\rho+2\rho\,\zeta_3^2\qquad\qquad \rho\in(-1,0)\cup(0,+1).$$

Proof of (i):
For $\epsilon=0$ we have

$$g=\left(\frac{\alpha}{u}+c_2+c_1\,u+c_0\,u^2\right)\,g_0\qquad\qquad g_0=du^2+u^2\,d\phi^2$$

Taking $u\in(0,+\infty)$ and ϕ azimuthal g_0 is just the canonical flat metric on $\mathbb{R}^2$. Going back to cartesian coordinates we have $u=\sqrt{x^2+y^2}$ therefore we must take $\alpha=c_1=0$. The potential $m\,x=m/u$ is not globally defined so we take $m=0$ and change $n\to m$. Then transforming back the observables to cartesian coordinates gives the desired result up to scalings. $\square$

Proof of (ii):
For $\epsilon=-1$ the change of coordinates $u=\tanh\chi$, up to an argument similar to the case $\epsilon=+1$ which gives $\alpha=c_1=0$, brings the metric into the form

$$g=\lambda\Big(1+\rho\,\cosh(2\chi)\Big)\,g(H^2,\mathrm{can})\qquad \lambda=\frac{c_2-c_0}{2}\qquad \rho=\frac{c_2+c_0}{c_2-c_0}$$

and we will write the conformal factor

where ρ has to be appropriately constrained to prevent the vanishing of the conformal factor. Again we must have $m=0$ and change $n\to m$. The final form of the quartic integral is obtained up to scalings. □

Proof of (iii):
For $\epsilon=1$ the change of coordinates $u=\tan\theta$ brings the metric into the form

$$g=\Big(\alpha\frac{\cos^3\theta}{\sin\theta}+\frac{c_2+c_0}{2}+\frac{c_1}{2}\,\sin(2\theta)+\frac{c_2-c_0}{2}\,\cos(2\theta)\Big)\,g(S^2,\mathrm{can})\qquad \theta\in(0,\pi).$$

One must impose $\alpha=0$ and since $\sin\theta$ is not differentiable at the poles $c_1=0$. The conformal factor becomes

$$\lambda\,\Big(1+\rho\,\cos(2\theta)\Big)\qquad \lambda=\frac{c_2+c_0}{2}\qquad \rho=\frac{c_2-c_0}{c_2+c_0}$$

and the parameter ρ is constrained in order to avoid a zero which would induce a curvature singularity. The final form of the observables, setting $m=0$, changing $n\to m$ and some scalings is then easily obtained. □

In all of the following Sections we will set $r\neq 0$.

4.2 Second case: $\alpha=c_1=0$

In this case $p=c_2\,x^2+c_0$ and changing $d\;\to\;\delta$ we have

$$F=x^4+(2\delta-4\,r\,c_2)x^2+\delta^2-4\,r\;c_0\qquad\qquad \beta^2=\frac{1}{2\,r}(x^2+\delta-\sqrt{F}).\tag{73}$$

Defining

$$a+\widetilde{a}=2\delta-4\,r\,c_2\qquad\qquad a\,\widetilde{a}=\delta^2-4\,r\,c_0$$

we have

$$F=(x^2+a)(x^2+\widetilde{a})\qquad\qquad 2r\,G=\sqrt{F}-x^2-\frac{a+\widetilde{a}}{2}$$

and we can state:

Theorem 8 *The integrable system described by Theorem 3:*

(i) Is trivial for $\widetilde{a}=a\in\mathbb{R}$.

(ii) Is not defined on a manifold if $\min(a,\widetilde{a})<0$ *and* $a\neq\widetilde{a}$.

(iii) Is defined on $\mathbb{H}^2$ *if* $\widetilde{a}=0$ *and* $a>0$. *It can be written*

$$\left\{\begin{array}{l} 2\,H=\dfrac{\epsilon}{B}\left(P_v^2+\dfrac{c}{s^2}\,P_\phi^2+\kappa\,\dfrac{s}{(c+1)^2}\,\cos\phi-\kappa\,l\,\dfrac{s^2}{(c+1)^2}+m\right)\qquad \epsilon=\pm1\\[2ex] Q=P_\phi^4+4\,\epsilon\,H\,P_\phi^2-4\kappa\Big(\sin\phi\,P_v+\dfrac{\cos\phi}{s}\,P_\phi\Big)P_\phi+4\kappa\,l\,P_\phi^2+\kappa^2\,\dfrac{s^2}{(c+1)^2}\,\sin^2\phi\end{array}\right.\tag{74}$$

with

$$B=\delta-\frac{1}{c+1}.$$

For $\epsilon=-1$ and $\delta=0$ the manifold becomes $\mathbb{S}^2$ and the integrable system is Kovalevskaya's given by (5) and (7).

(iv) Is defined on $\mathbb{S}^2$ if $0<\widetilde{a}<a$. It can be written

$$\left\{\begin{array}{l} 2\,H=\frac{\epsilon}{B}\left(P_v^2+\frac{D}{S^2}\,P_\phi^2+\kappa\,k^4\,\frac{SC}{(D+1)^2}\,\cos\phi-\kappa\,l\,k^4\,\frac{S^2}{(D+1)^2}+m\right)\qquad \epsilon=\pm 1\quad k^2\in(0,1)\\ Q=P_\phi^4+4\,\epsilon\,H\,P_\phi^2-4\kappa\Big(\sin\phi\,P_v+\frac{C}{S}\,\cos\phi\,P_\phi\Big)P_\phi+4\kappa\,l\,P_\phi^2+\kappa^2\,k^4\,\frac{S^2}{(D+1)^2}\,\sin^2\phi \end{array}\right. \tag{75}$$

with

$$B=\delta-1+\frac{k^2}{D+1}.$$

(v) Is defined on $\mathbb{S}^2$ if $a\in\mathbb{C}\backslash\{0\}$ and $\widetilde{a}=\overline{a}$. It can be written

$$\left\{\begin{array}{l} 2\,H=\frac{\epsilon}{B}\left(P_v^2+\frac{\sigma}{S^2}\,P_\phi^2-\kappa\,k^2\,k'^2\,\frac{SC}{D^3}\,\cos\phi+\kappa\,k^2\,k'^2\,\frac{S^2}{D^2}+m\right)\qquad \epsilon=\pm 1\quad k^2\in(0,1)\\ Q=P_\phi^4+4\,\epsilon\,H\,P_\phi^2-\kappa\left(\sin\phi\,P_v+\frac{C}{SD}\,\cos\phi\,P_\phi\right)P_\phi+\kappa\,l\,P_\phi^2-\kappa^2\,k^2\,k'^2\,\frac{S^2}{4\,D^2}\,\sin^2\phi \end{array}\right. \tag{76}$$

with

$$B=\delta+1-2\frac{k'^2}{D^2}.$$

We use again the shorthand notations for hyperbolic and elliptic functions given in Theorem 4.

Proof of (i):
In this case $G\equiv 0$ hence Q_4 is reducible and the system is trivial. $\Box$

Proof of (ii):
Taking $a<\widetilde{a}$ we have to consider two cases:

$$(1):\quad a=-x_2^2\qquad \widetilde{a}=-x_1^2\quad 0<x_1<x_2\qquad\qquad (2):\quad a=-x_2^2<0\qquad \widetilde{a}=x_1^2.$$

In both cases positivity allows $x\in(x_2,+\infty)$. If $\beta^2(x_2)\neq 0$, as already seen in the proof of Theorem 4, case (ii), $x=x_2$ remains a true singularity. If $\beta^2(x_2)=0$ then $x=x_2$ remains a curvature singularity since

$$R=\frac{r\,x_2}{2(x-x_2)}+O(1).$$

In the first case we may also take $x\in(-x_1,+x_1)$. If $\beta^2(x_1)\neq 0$, as already seen in the proof of Theorem 4, case (ii), $x=x_1$ remains a true singularity. If $\beta^2(x_1)=0$ the curvature is

$$R=\frac{r\,x_1}{2(x-x_1)}+O(1)$$

and $x=x_1$ remains a curvature singularity $\Box$

Proof of (iii):

Defining $x = 1/u$ and setting $\epsilon = \pm 1$ for the sign of r, the metric is homothetic to

$$g = \epsilon\, B\left(\frac{du^2}{1+a\,u^2} + \frac{u^2\,d\phi^2}{\sqrt{1+a\,u^2}}\right) \qquad B = \delta - \frac{1}{1+\sqrt{1+a\,u^2}} \qquad a = \pm 1$$

For $a = -1$ the curvature

$$R = -\frac{3}{4}\,\frac{1}{\delta - 1}\,\frac{1}{1-u} + O\left(\frac{1}{\sqrt{1-u}}\right)$$

is singular for $u \to 1-$ and it remains singular for $\delta = 1$ since in this case we have

$$R = -\frac{1}{4}\,\frac{1}{1-u} + O(1).$$

So we must consider $a = +1$ and the change of variable $u = \sinh v$ gives for the metric

$$g = \epsilon\left(\delta - \frac{1}{c+1}\right)\left(dv^2 + \frac{s^2}{c}\,d\phi^2\right) \qquad v \in (0, +\infty) \qquad \phi \text{ azimuthal}$$

using the notations of Theorem 4, case (iii).

If the conformal factor never vanishes for $v \geq 0$ we will get for manifold $M = \mathbb{H}^2$. We have therefore two cases:

$$\left(\epsilon = +1 \qquad \delta > \frac{1}{2} \quad \Rightarrow \quad B > 0\right) \qquad \text{or} \qquad \left(\epsilon = -1 \qquad \delta < 0 \quad \Rightarrow \quad B < 0\right). \tag{77}$$

In these two cases the global definiteness proof needs not be repeated since it follows closely Theorem 4, case (iii).

A very unusual manifold bifurcation appears for $\epsilon = -1$ and $\delta = 0$. The volume is *finite* with a $C^\infty([0, +\infty))$ curvature. To understand what is the true manifold, let us make the coordinate change $\sinh(v/2) = \tan\theta$ mapping $v \in (0, +\infty) \to \theta \in (0, \pi/2)$. The metric becomes

$$g = 2\left(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta}\,d\phi^2\right) \qquad\qquad R = 2\frac{(2-\sin^2\theta)}{(1+\sin^2\theta)^2}$$

which would be a metric on $P^2(\mathbb{R})$. However we can extend the range of θ to $(0, \pi)$ and the points $\theta = 0$ and $\theta = \pi$ are just the apparent coordinate singularities giving for manifold $\mathbb{S}^2$. We recognize this metric as being Kovalevskaya's and a simple computation shows that H and Q are indeed given by (5) and (7) since one must set $l = m = 0$ to avoid the singularity on the "equator" $\theta = \pi/2$. $\square$

Proof of (iv) and (v):

Here too there is no need to repeat the proofs given in Theorem 4, cases (iv) and (v), provided that the conformal factors do not vanish for $v \in [0, 2K]$. The resulting constraints for the case (iv) are

$$\left(\epsilon = +1 \qquad \delta > 1 - \frac{k^2}{2} \quad \Rightarrow \quad B > 0\right) \qquad \text{or} \qquad \left(\epsilon = -1 \qquad \delta < k' \quad \Rightarrow \quad B < 0\right) \tag{78}$$

and for the case (v)

$$\left(\epsilon = +1 \qquad \delta > 1 \quad \Rightarrow \quad B > 0\right) \qquad \text{or} \qquad \left(\epsilon = -1 \qquad \delta < 1 - 2k^2 \quad \Rightarrow \quad B < 0\right). \tag{79}$$

They are easily obtained since B is monotonous. $\square$

4.3 Third case: $\alpha \neq 0$

We have for F the structure

$$F \equiv (x^2+d)^2 - 4r\,p = x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0 \qquad a_3 = -4r\alpha. \tag{80}$$

Since $a_3 \neq 0$ no matter whether $a_1 = -4r\,c_1$ vanishes or not F is the most general quartic polynomial.

The relevant functions are

$$\beta^2 = \frac{1}{2r}\Big(d + x^2 - \sqrt{F}\Big) \qquad G = \frac{1}{4r}\Big(-a_2 + \frac{a_3^2}{4} - a_3\,x - 2\,x^2 + 2\sqrt{F}\Big) \tag{81}$$

and the metric

$$g = \beta^2\left(\frac{dx^2}{F} + \frac{d\phi^2}{\sqrt{F}}\right). \tag{82}$$

Let us begin with two preparatory lemmas:

Lemma 1 *The points $x = \pm\infty$ are apparent singularities of the metric (82). The metric may be defined on a manifold for $x \in (x_0, +\infty)$ with $F(x_0) = 0$ and provided that F be strictly positive on this interval. A simple zero or a fourfold zero of F are excluded.*

Proof:
For $x \to +\infty$ we have

$$g \sim \frac{-a_3}{r}\left(\frac{dx^2}{4x^3} + \frac{d\phi^2}{4x}\right) = \frac{-a_3}{r}\Big(d\tau^2 + \tau^2\,d\Phi^2\Big) \qquad \tau = \frac{1}{\sqrt{x}} \to 0+ \qquad \Phi = \frac{\phi}{2}$$

and if we take Φ to be azimuthal this singularity is apparent.

For $x \to -\infty$ we have

$$g \sim \frac{-a_3}{r}\left(\frac{dx^2}{4x^3} + \frac{d\phi^2}{4x}\right) = \frac{a_3}{r}\Big(d\tau^2 + \tau^2\,d\Phi^2\Big) \qquad \tau = \frac{1}{\sqrt{-x}} \to 0+$$

and we are led to the same conclusion. The change of sign of the metric from $x \to +\infty$ to $x \to -\infty$ forbids $x \in \mathbb{R}$, hence there must be a zero x_0 for which F vanishes but remains strictly positive in $(x_0, +\infty)$.

Let it be supposed that the zero of F is simple. We can write

$$F = u\,P(u) \qquad u \equiv x - x_0 \qquad P(u) = u^3 + b_2 u^2 + b_1 u + b_0 \qquad b_0 > 0.$$

It follows that the product of the roots of P is positive hence we will have at least one positive root, which is to be excluded.

The case where the zero is fourfold is also excluded since then G vanishes identically. $\square$

This lemma covers the case of an infinite range for x. However positivity allows also a finite range of x between two zeroes of F. This is considered in

Lemma 2 *If F is strictly positive between two of its zeroes (x_1, x_2) the metric (82) may be defined on a manifold only if x_1 and x_2 are twofold.*

Proof:

Let $x=x_1$ and $x=x_2>x_1$ be simple zeroes of F. We have

$$F(x)=(x-x_1)(x_2-x)\,P(x) \qquad \beta^2=\frac{1}{2r}\,B \qquad B=d+x^2-\sqrt{F}$$

where $P(x)$ must be strictly positive for $x\in\,[x_1,x_2]$. The curvature, omitting the factor $1/2r$ in β^2 is singular at both end-points:

$$R=\frac{3}{8}\,\frac{(x_1-x_2)}{(d+x_1^2)}\,\frac{P(x_1)}{x-x_1}+O(1) \qquad R=-\frac{3}{8}\,\frac{(x_1-x_2)}{(d+x_2^2)}\,\frac{P(x_2)}{x-x_2}+O(1)$$

provided that B does not vanish in $[x_1,\,x_2]$. If $B(x_1)=0$ the curvature remains singular

$$R=\frac{x_1}{4}\frac{1}{x-x1}+O(1)$$

for $x=x_1$.

Alternatively we can take x_1 to be twofold $F=(x-x_1)^2(x_2-x)P(x)$ with $P>0$ for $x\in\,[x_1,x_2]$ in which case the curvature becomes continuous at $x=x_1$ but remains singular at $x=x_2$ for all possible values of d. However for the special case $d=x_2=0$ the singularity for $x=x_2$ disappears leaving a continuous curvature. This special case is therefore given by

$$F=(x-x_1)^2\,(x^2-b\,x) \qquad d=x_2=0 \qquad b>0 \qquad x\in\,(x_1,0)$$

with the metric

$$\mathcal{B}\left(\frac{dx^2}{(x-x_1)^2(b-x)\sqrt{-x}}+\frac{d\phi^2}{(x-x_1)\sqrt{b-x}}\right) \qquad \mathcal{B}=(-x)^{3/2}-(x-x_1)\sqrt{b-x}$$

but one has

$$\mathcal{B}(x_1)=(-x_1)^{3/2}>0 \qquad \text{and} \qquad \mathcal{B}(0)=-\sqrt{b}\,(-x_1)<0$$

showing that $\mathcal{B}(x)$ vanishes in $(x_1,0)$ leading to a curvature singularity.

The last possible case is $F(x)=(x-x_1)^2(x_2-x)^2$ giving a continuous curvature at both end-points. $\square$

These results will be used to prove:

Theorem 9 *The locally integrable system described in Theorem 3:*

(i) Is globally defined on $M=\mathbb{H}^2$ if F has a twofold zero and $\delta\neq 0$. Its hamiltonian is

$$2\,H=\frac{1}{B}\left(F\,P_v^2+\sqrt{F}\,\frac{P_\Phi^2}{v^2}+\kappa\sqrt{F}\,v^2\,G_{,v^2}\,\cos(2\Phi)+l\,v^2\,G+m+n\,v^2\right) \tag{83}$$

where $v\in\,(0,+\infty)$ and Φ is azimuthal, with

$$F=1+2\sigma\,v^2+v^4 \qquad \sigma\in\,(-1,+1) \qquad G=\frac{\sqrt{F}-1-\sigma\,v^2}{v^4}-\frac{(1-\sigma^2)}{2}$$

and

$$B=\rho+\delta\,v^2-\frac{\sqrt{F}-1-\sigma\,v^2}{v^2} \qquad \mathcal{M}=\sin\Phi\,\sqrt{F}\,P_v+\frac{B}{\rho}\,\cos\Phi\,\frac{P_\Phi}{v}.$$

The quartic integral is

$$\begin{aligned}Q=&\frac{1}{4}\,P_\Phi^4+H\,P_\Phi^2+\kappa\frac{\rho}{B}\,\mathcal{M}^2+\kappa\left[\sigma+\frac{1}{v^2}\left(2-\frac{B}{\rho}-\frac{\rho}{B}\sqrt{F}\right)\right]\cos^2\Phi\,P_\Phi^2\\&-\kappa\left[\frac{\sigma}{2}+\frac{1}{v^2}\left(1-\frac{\rho}{B}\sqrt{F}\right)\right]\,P_\Phi^2+\frac{l}{2}\,P_\Phi^2+\frac{\kappa^2}{2}\left(\frac{\rho}{B}\,\sqrt{F}\,v^2\,G_{,v^2}-G\right)\,\sin^2(2\Phi)\\&+\kappa\,\frac{\rho}{B}\left(-\kappa\,\sqrt{F}\,v^2\,G_{,v^2}+l\,v^2\,G+m\frac{\rho-B}{\rho}+n\,v^2\right)\,\sin^2\Phi\end{aligned} \tag{84}$$

The parameters ρ and δ must ensure that $B\neq 0$ for $v\geq 0$.

(ii) Is globally defined on $M=\mathbb{S}^2$ if F has a twofold zero and $\delta=0$. Its hamiltonian is

$$2H=\frac{1}{B}\left(P_v^2+\mu\,\frac{P_\Phi^2}{S^2}+2\kappa\,k^2\,k'^2\frac{(D^2-2k'^2)}{D^4}\,S^2\,\cos(2\Phi)+m\right) \tag{85}$$

with

$$\mu=1-k^2\frac{S^2\,C^2}{D^2} \qquad\qquad B=\rho-2k^2\,k'^2\,\frac{S^2}{D^2}$$

and the quartic integral

$$\begin{aligned} Q=&\frac{1}{4}\,P_\Phi^4+H\,P_\Phi^2+\kappa\frac{\rho}{B}\,\mathcal{L}^2-2\kappa\frac{k^2\,k'^2}{B}\left(\rho+1-2k^2+2\frac{k^2\,k'^2}{\rho}\frac{C^2}{D^4}\right)\frac{S^2}{D^2}\cos^2\Phi\,P_\Phi^2\\ &-\frac{1}{B}\left((k^2-3/2)B+2\rho\,\frac{k'^2}{D^2}+2k^2k'^2\frac{C^2-S^2\,D^2}{D^4}\right)P_\Phi^2\\ &-2\kappa^2(k^2k'^2)^2\left(\frac{\rho}{2}-k^2+\frac{k'^2}{D^2}\right)\frac{S^4}{D^4}\,\sin^2(2\Phi)+2\kappa\frac{k^2k'^2}{B}\left(m-\rho\kappa\frac{(D^2-2k'^2)}{D^2}\right)\frac{S^2}{D^2}\sin^2\Phi \end{aligned} \tag{86}$$

with

$$\mathcal{L}=\sin\Phi\,P_v+\frac{B}{\rho}\,\frac{C}{SD}\cos\Phi\,P_\Phi$$

The parameters ρ and δ must ensure that $B\neq 0$ for $v\in\,[0,2K]$.

(iii) Is globally defined in $\mathbb{H}^2$ if F has a threefold zero and $\delta\neq 0$. Its hamiltonian is

$$2H=\frac{1}{B}\left(P_v^2+\frac{c}{s^2}P_\Phi^2+\kappa\,\frac{s^2}{(c+1)^3}\,\cos(2\Phi)+l\,\frac{(c+3)s^4}{(c+1)^3}+m\,s^2\right) \tag{87}$$

and the quartic integral

$$\begin{aligned} Q=&\frac{1}{4}P_\Phi^4+H\,P_\Phi^2+\frac{2\rho\kappa}{B}\,\mathcal{M}^2+\frac{2\kappa}{B}\left(\frac{\rho}{c+1}-\Delta-\frac{B}{4}\right)P_\Phi^2+4\,l\,P_\Phi^2\\ &+\frac{2\kappa}{B}\left(\frac{\Delta}{c+1}-\frac{\Delta^2}{\rho}+\frac{B}{2(c+1)^2}\right)s^2\,\cos^2\Phi\,P_\Phi^2-\frac{\kappa^2}{B(c+1)^3}\left(\Delta+\frac{B}{4(c+1)}\right)s^4\,\sin^2(2\Phi)\\ &+2\frac{\rho\kappa}{B}\left(-\frac{\kappa}{(c+1)^3}+l\frac{(c+3)s^2}{(c+1)^3}+m\right)s^2\,\sin^2\Phi \end{aligned} \tag{88}$$

with

$$B=\rho+\Delta\,s^2 \qquad\qquad \Delta=\delta+\frac{1}{2(c+1)^2} \qquad\qquad \mathcal{M}=\sin\Phi\,P_v+\frac{B}{\rho}\,\cos\Phi\,\frac{P_\Phi}{s}.$$

The parameters ρ and δ must ensure that $B\neq 0$ for $v\geq 0$.

(iv) Is globally defined in $\mathbb{S}^2$ if F has a threefold zero and $\delta=0$ and $\rho=-1/2$. Its hamiltonian is

$$2H=P_\theta^2+\frac{(1+\sin^2\theta)}{\sin^2\theta}\,P_\Phi^2+\kappa\,\sin^2\theta\,\cos(2\Phi) \tag{89}$$

and its quartic integral

$$\begin{aligned} Q=P_\Phi^4-2H\,P_\Phi^2+\kappa\,\mathcal{L}^2+\kappa\Big(1+\cos^2\theta(1-\sin^2\theta\,\cos^2\Phi)\Big)\,P_\Phi^2&\\ +\kappa^2\sin^2\theta\,\sin^2\Phi(\sin^2\theta\,\cos^2\Phi-1)& \end{aligned} \tag{90}$$

with

$$\mathcal{L}=\sin\Phi\,P_\theta+\cos^2\theta\,\frac{\cos\Phi}{\tan\theta}\,P_\Phi$$

which is nothing but a special case of Goryachev system.

Proof of (i):
F has a twofold zero for $x = x_1$. Using $u = x - x_1$ we can write $F = u^2(u^2 + b_1 u + b_0)$ with $b_0 > 0$ and $u \in (0, +\infty)$. We will use as a new variable $u = \sqrt{b_0}/v^2$ with $v \in (0, +\infty)$ which results in

$$\sqrt{F} = b_0 \frac{\sqrt{\Psi}}{v^4} \qquad \Psi(v^2) = 1 + 2\sigma\, v^2 + v^4 \qquad \sigma = \frac{b_1}{2\sqrt{b_0}}$$

$$\beta^2 = \frac{b_0}{2r}\,\frac{B}{v^2} \qquad B(v^2) = \rho + \delta\, v^2 - \frac{\sqrt{\Psi} - 1 - \sigma\, v^2}{v^2}$$

$$G = \frac{b_0}{2r}\,\Gamma \qquad \Gamma = \frac{\sqrt{\Psi} - 1 - \sigma\, v^2}{v^4} - \frac{(1 - \sigma^2)}{2}$$

with

$$\delta = \frac{d + x_1^2}{b_0} \qquad \rho = \frac{2\,x_1}{\sqrt{b_0}} - \sigma \qquad \sigma \in (-1, +1).$$

The parameter m must vanish and this allows the change $n \to m$. Transforming the observables given by Theorem 3 (up to various scalings and leaving aside a reducible piece proportional to the hamiltonian) one obtains (83) and (84) up to the substitutions $\Psi \to F$ and $\Gamma \to G$.

For the global aspects, let us define the coordinate $t \in (0, 1)$ by

$$t \equiv \tanh\frac{\chi}{2} = \frac{v}{1 + \sqrt{v^2 + 1}}\, e^{\eta(v)} \qquad \eta(v) = -\int_v^{+\infty} \left(\frac{1}{\sqrt[4]{F(x)}} - \frac{1}{\sqrt{1 + x^2}} \right) \frac{dx}{x} \tag{91}$$

giving for the metric

$$g = \Omega^2\, g(S^2, \mathrm{can}) \qquad \Omega^2 = \frac{v^2\, B}{\sqrt{F}\, \sinh^2\chi} \tag{92}$$

and for the hamiltonian

$$2H = \frac{1}{\Omega^2}\Big(M_1^2 + M_2^2 - M_3^2 \Big) + \kappa\, \Lambda_1\, (\eta_1)^2 + \Lambda_2 \tag{93}$$

with

$$\Lambda_1 = 2\frac{v^2\, \sqrt{F}\, G_{,v^2}}{\sinh^2\chi} \qquad \Lambda_2 = (-\kappa\, \sqrt{F}\, G_{,v^2} + l\, G + m) v^2.$$

All the functions involved do depend on v^2 hence on t^2 which is globally defined. They are C^∞ for $t^2 \in (0, 1)$. To extend this result to $[0, 1)$ we just need to check their behaviour for $t \to 0+$ using

$$\frac{v}{2} = \tau + \sigma\, \tau^3 + \tau^5 + O(\tau^7) \qquad \tau = e^{-\eta(0)}\, t.$$

The relation

$$\mathcal{M} = \frac{2t}{v}\,\frac{F^{1/4}}{1 - t^2} \left\{ M_1 + \frac{(1 - t^2)^2}{4t^2} \left(\frac{B}{\rho\, F^{1/4}} - \frac{1 + t^2}{1 - t^2} \right) \eta_1\, M_3 \right\}$$

allows to check that $\mathcal{M}$ is globally defined, and writing the quartic integral as

$$Q = \frac{1}{4}\, M_3^4 + H\, M_3^2 + \kappa \frac{\rho}{B} \mathcal{M}^2 + \Lambda_3\, \eta_1^2\, M_3^2 + \Lambda_4\, M_3^2 + \Lambda_5\, (\eta_1\, \eta_2)^2 + \Lambda_6\, \eta_2^2$$

where the Λ_i are easily retrieved from (84). Elementary checks for $t \to 0$ show that it is indeed globally defined.

To discuss the sign of B it is convenient to use, instead of $v \in [0, +\infty)$, the new variable w defined by

$$\mathrm{cn}(2w) = \frac{1 - v^2}{1 + v^2} \qquad \sigma = 1 - 2k^2 \qquad w \in [0, K).$$

Then we have

$$B-\rho=\delta\,\frac{\mathrm{sn}^2 w\,\mathrm{dn}^2 w}{\mathrm{cn}^2 w}-2k^2k'^2\,\frac{\mathrm{sn}^2 w}{\mathrm{dn}^2 w}\qquad \delta\neq 0.$$

Defining x for new variable

$$x=x_0\,\mathrm{cn}^2 w\qquad\qquad x_0=\frac{k^2}{k'^2}\qquad\qquad x\in\ (0,x_0]$$

we can write

$$B-\rho=\delta\,k'^2\,\frac{(x_0-x)(x-r_-)(x-r_+)}{x(x+1)}\qquad\qquad r_\pm=\frac{1-\delta\pm\sqrt{1-2\delta}}{\delta}$$

which explicitly exhibits its three zeros.

For $\delta\geq 1/2$ the roots $r_\pm$ are complex hence $B-\rho\geq 0$ so that if $\rho>0$ then $B>0$ for all $x\in\ (0,x_0]$.

For $\delta<0$ the roots $r_\pm$ are both negative in which case $B-\rho\leq 0$ will imply that for $\rho<0$ we have $B>0$ for all $x\in\ (0,x_0]$.

The last case is $\delta\in\ (0,1/2)$ for which we have $0<r_-<r_+$. If $r_-\geq x_0$ then $B-\rho\geq 0$ and then $\rho>0$ implies again that $B>0$ for all $x\in\ (0,x_0]$. In the remaining cases either $r_-<x_0\leq r_+$ (then $B'(x)$ has a single simple root $0<x_1<x_0$) or $r_-<r_+<x_0$ (then $B'(x)$ has two simple roots with $0<x_1<x_2<x_0$), the minimum value of $B-\rho$ is always given by $B(x_1)-\rho$ where x_1 is the smallest positive root of $B'(x)$ and if $\rho>B(x_1)$ then $B>0$ for all $x\in\ (0,x_0]$.

Taking into account the relation

$$D_x\,B=-\frac{\delta\,k'^2}{(x+1)^2}\,R(x)\qquad R(x)=x^2+2x-(x_0+1)(r_++r_-+1)+\frac{2x_0}{x}+\frac{x_0}{x^2}$$

we see that, in practice, to get x_1 one must solve for the smallest positive root of the quartic equation $R(x)=0$. $\square$

Proof of (ii):

Let us start from the hamiltonian given in (83) with $\delta=0$ and in which we change $v\to u$. It is then possible to introduce as a new variable v defined by

$$\mathrm{cn}\,(2v)=\frac{1-u^2}{1+u^2}\qquad\qquad \sigma=k'^2-k^2\qquad v\in\ (0,K).$$

The metric

$$g=B\left(dv^2+\frac{S^2}{\mu}\,d\Phi^2\right)\qquad B=\rho-2k^2k'^2\frac{S^2}{D^2}\qquad \mu=1-k^2\frac{S^2\,C^2}{D^2}$$

is now multiplied by a *finite* conformal factor B.

Since Φ is azimuthal we can extend v to $(0,2K)$. The points $v=0$ and $v=2K$ are apparent singularities and the manifold is $M=\mathbb{S}^2$. Computing G leads to the hamiltonian given in (85). Since the potentials

$$l\,u^2G+n\,u^2=2l\,k^2k'^2\,\frac{(k^2\,D^2-k'^2)S^4}{C^2D^2}+n\,\frac{S^2\,D^2}{C^2}$$

are singular at the "equator" $v=K$ we must set $l=n=0$. Similar transformations in Q given by (84) lead eventually to (86).

To ascertain the global structure we will define

$$t\equiv\tan\frac{\theta}{2}=\frac{\mathrm{sn}\,v}{\mathrm{cn}\,v+\mathrm{dn}\,v}\,e^{\eta(v)}\qquad\qquad \eta(v)=\int_0^v\frac{\sqrt{\mu(x)}-1}{\mathrm{sn}\,x}\,dx$$

which gives for the metric

$$g=\Omega^2\,g(S^2,\mathrm{can})\qquad\qquad \Omega^2=\frac{B}{\mu}\frac{(1+t^2)^2\,S^2}{4t^2}.$$

The hamiltonian becomes

$$2H=\frac{1}{\Omega^2}(L_1^2+L_2^2+L_3^2)+\Lambda_1\,(\zeta_2^2-\zeta_1^2)+\frac{m}{B}$$

with

$$\Lambda_1=2\kappa\,k^2k'^2\,\frac{(D^2-2k'^2)}{D^4}\frac{(1+t^2)^2\,S^2}{4t^2}.$$

Obviously the functions Ω^2, Λ_1 and B are C^∞ functions of $t^2\in(0,1]$ and using the series expansion

$$v=2\tau+\frac{2}{3}\,(2k^2-1)\,\tau^3+O(\tau^5)\qquad\qquad \tau=e^{-\eta(K)}\,\frac{t}{k'}\tag{94}$$

this remains true for $t^2\in[0,1]$, i. e. around the north pole ($\zeta_3=+1$). There is no need to check for the south pole: $t^2\to+\infty$ or $\zeta_3=-1$ due to the relations

$$t(2K-v)=\frac{1}{t(v)}\qquad\Longleftrightarrow\qquad v(1/t)=2K-v(t).\tag{95}$$

The relation

$$\mathcal{L}=\frac{2t\,\sqrt{\mu}}{S(1+t^2)}\left\{L_1+\frac{1}{4t^2}\left[\frac{BC}{\rho\,D}(1+t^2)-(1-t^2)\right]\zeta_2\,L_3\right\}$$

allows to check that $\mathcal{L}$ is globally defined, and writing the quartic integral as

$$Q=\frac{1}{4}\,L_3^2+H\,L_3^2+\frac{\kappa\rho}{B}\,\mathcal{L}^2+\Lambda_2\,\zeta_2^2\,L_3^2+\Lambda_3\,L_3^2+\Lambda_4\,\zeta_1^2\,\zeta_2^2+\Lambda_5\,\zeta_2^2$$

where the Λ_i are easily retrieved from (86), elementary computations for $t^2\to 0$ show that it is indeed globally defined.

Imposing the non-vanishing of the conformal factor B for $v\in[0,2K]$ gives

$$\Big(\ B>0\quad\Longleftrightarrow\quad\rho>0\ \Big)\qquad\qquad\Big(\ B<0\quad\Longleftrightarrow\quad\rho<2k^2\Big)$$

after some elementary computations. $\square$

Proof of (iii):
F has a threefold zero at $x=x_1$, so we will write $F=(x-x_1)^3(x-x_1+b)$ with $b>0$. Since $x\in(x_1,+\infty)$ it is convenient to define

$$s\equiv\sinh v=\sqrt{\frac{b}{x-x_1}}\qquad\qquad c\equiv\cosh v\qquad v\in(0,+\infty)$$

It follows that

$$\sqrt{F}=b^2\,\frac{c}{s^4}\qquad\qquad \beta^2=\frac{b^2}{2r}\frac{B}{s^2}\qquad\qquad G=\frac{b^2}{16r}\,\frac{(c+3)\,s^2}{(c+1)^3}$$

where
$$\rho=-\frac{b-4x_1}{2b}\qquad\qquad \delta=\frac{d+x_1^2}{b^2}\qquad\qquad B=\rho+\delta\,s^2+\frac{s^2}{2(c+1)^2}$$

The parameter m must again vanish and this allows the change $n\to m$. Transforming the observables given by Theorem 3 (up to various scalings) one obtains (87) and (88).

To study the global aspects we need again the relations (40)

$$t\equiv\tanh\frac{\chi}{2}=\tanh\frac{v}{2}\,e^{\eta(v)}\qquad\qquad \eta(v)=-\int_v^{+\infty}\frac{(\sqrt{\cosh x}-1)}{\sinh x}\,dx$$

mapping $v\in(0,+\infty)\to t\in(0,1)$. We get this time

$$g=\Omega^2\,g(H^2,\mathrm{can})\qquad\qquad \Omega^2(t^2)=B\,\frac{(1-t^2)^2\,\sinh^2 v}{4t^2\,\cosh v}\tag{96}$$

The relations (41) and (42), proved for Theorem 4, case (iii), are still valid

$$\tanh\frac{v}{2}=t\,e^{-\psi(t^2)}\qquad\qquad v=\tau-\frac{\tau^3}{24}+O(\tau^5)\qquad \tau=2e^{-\eta(0)}\,t.$$

The hamiltonian can be written

$$2H=\frac{1}{\Omega^2(t^2)}\Big(M_1^2+M_2^2-M_3^2\Big)+\kappa\,\Lambda_1(t^2)\,\eta_1^2+\Lambda_2(t^2)\tag{97}$$

with
$$\Lambda_1(t^2)=\frac{\kappa}{B}\frac{(1-t^2)^2}{2t^2}\frac{s^2}{(c+1)^3}\qquad\quad \Lambda_2(t^2)=\frac{s^2}{B}\Big(\frac{-\kappa+l(c+3)s^2}{(c+1)^3}+m\Big)$$

and is therefore globally defined on $\mathbb{S}^2$ while the quartic integral

$$Q=\frac{1}{4}\,M_3^4+H\,M_3^2+2\frac{\rho\kappa}{B}\mathcal{M}^2+\Lambda_3(t^2)M_3^2+\kappa\,\Lambda_4(t^2)\,\eta_1^2\,M_3^2-\kappa^2\,\Lambda_5(t^2)\,\eta_1^4+\kappa\,\Lambda_6(t^2)\,\eta_2^2$$

and
$$\mathcal{M}=2\frac{\sqrt{c}}{1-t^2}\frac{t}{s}\left[M_1+\frac{(1-t^2)^2}{4t^2}\left(\frac{B}{\rho\sqrt{c}}-\frac{(1+t^2)}{(1-t^2)}\right)\zeta_1\,M_3\right]$$

are also globally defined.

As a last step one has to impose the non-vanishing of B for $v\in[0,+\infty)$. The resulting constraints on the parameters are

$$B>0:\qquad \delta>0\ \wedge\ \rho>0$$

$$B<0:\qquad \Big(\delta\le-\frac{1}{8}\ \wedge\ \rho<0\Big)\ \vee\ \Big(\delta\in(-\frac{1}{8},0)\ \wedge\ \rho<|\delta|(x_0+1)(3x_0-1)-\frac{1}{2}\Big)$$

where the cubic
$$2|\delta|\,x_0(x_0+1)^2=1$$

has x_0 for unique real solution. $\square$

Proof of (iv):
For $\delta=0$ the conformal factor, given in (iii), is bounded and for $\rho=-1/2$ it exhibits a strong decrease for $v\to+\infty$. The metric

$$g=-\left(\frac{dv^2}{c+1}+\frac{s^2}{c(c+1)}\,d\Phi^2\right)$$

under the change of variable

$$\theta=\arctan\Big(\sinh\frac{v}{2}\Big)\qquad\qquad v\in\,(0,+\infty)\quad\longrightarrow\quad\theta\in\,(0,\,\pi/2)$$

becomes

$$g=-2\Big(d\theta^2+\frac{\sin^2\theta}{1+\sin^2\theta}\,d\Phi^2\Big)$$

on which it is clear that the range of θ can be extended to $(0,\pi)$ giving for manifold $M=\mathbb{S}^2$. So we recover Kovalevskaya's metric (6), but with a different potential as can be seen from the hamiltonian (89) and its quartic integral (90). In fact Goryachev [1] derived a generalization of Kovalevskaya with two more parameters. His potential, as quoted in [2], can be written

$$k\,\sin\theta\,\cos\phi+\sin^2\theta\Big(B_1\,\sin(2\Phi)+B_2\,\cos(2\Phi)\Big)$$

and we only got the special case $k=B_1=0$.

Writing the full system

$$\begin{aligned}&2H=L_1^2+L_2^2+2\,L_3^2+\kappa(\zeta_1^2-\zeta_2^2)\\&Q=L_3^4-2H\,L_3^2+\kappa\,\mathcal{L}^2+\kappa\Big[1+\zeta_3^2(1-\zeta_1^2)\Big]L_3^2+\kappa^2\,\zeta_2^2\,\Big(\zeta_1^2-1\Big)\end{aligned}\tag{98}$$

with $\mathcal{L}=L_1-\zeta_1\,\zeta_3\,L_3$ shows explicitly that it is globally defined. □

Let us now examine the last possible case where $x_1<x<x_2$ and $F=(x-x_1)^2(x_2-x)^2$. Let us prove

Theorem 10 *The locally integrable system defined in Theorem 3 becomes globally defined on $M=\mathbb{H}^2$. Its hamiltonian becomes*

$$2H=\frac{1}{B}\left(P_v^2+\frac{P_\phi^2}{\cosh^2 v}+\kappa\,\frac{\tanh v}{\cosh^2 v}\,\cos\phi+l\,\tanh^2 v+m\,\tanh v+n\right)\tag{99}$$

with

$$B(v)=\rho+2\sigma\,\tanh v+\tanh^2 v$$

and its quartic integral

$$\begin{aligned}Q=&\,P_\phi^4+2H\,(P_\phi^2-\kappa\sigma\,\cos\phi)-\kappa(\sin\phi\,P_v-\tanh v\,\cos\phi\,P_\phi)P_\phi-l\,P_\phi^2\\&-\frac{\kappa^2}{4}\frac{\sin^2\phi}{\cosh^2 v}+\frac{\kappa m}{2}\,\cos\phi\end{aligned}\tag{100}$$

where the conformal factor $B(v)$ should not vanish for $v\in\mathbb{R}$.

Proof :
The change of variable

$$v=\frac{1}{2}\ln\left(\frac{x-x_1}{x_2-x}\right)\qquad\qquad x\in\,(x_1,\,x_2)\to v\in\mathbb{R}$$

and the change of function

$$\beta^2=\frac{(x_2-x_1)^2}{4r}\,B(v)\qquad\qquad B(v)=\rho+2\sigma\,\tanh v+\tanh^2 v$$

turn the metric into

$$g=\frac{B}{r}(dv^2+\cosh^2 v\,d\phi^2).$$

The variable ϕ is no longer an azimuthal angle. Taking $(v,\,\phi)\in\mathbb{R}^2$, and provided that $B(v)$ does not vanish for $v\in\mathbb{R}$, we get a metric on $M=\mathbb{H}^2$ as discussed in the Appendix A. The function G becomes

$$G=\frac{(x_2-x_1)^2}{4r}\,\frac{1}{\cosh^2 v}$$

and up to a few scalings of the parameters, the observables H and Q are seen to be given by the formulas (99) and (100).

Let us first observe that the following functions

$$\phi=\sinh^{-1}\frac{\eta_1}{\sqrt{1+\eta_2^2}}\qquad\qquad \frac{1}{\cosh^2 v}=\frac{1}{1+\eta_2^2}\qquad\qquad \tanh v=\frac{\eta_2}{\sqrt{1+\eta_2^2}}$$

are globally defined. So writing the hamiltonian as

$$2H=\frac{1}{B}\left(M_1^2+M_2^2-M_3^2+\kappa\,\frac{\tanh v}{\cosh^2 v}\,\cos\phi+l\,\tanh^2 v+m\,\tanh v+n\right)$$

shows at once its global definiteness. For the quartic integral we have

$$Q=M_2^4+2H\,(M_2^2-\kappa\sigma\,\cos\phi)-\kappa(\sin\phi\,P_v-\tanh v\,\cos\phi\,M_2)M_2-l\,M_2^2$$
$$-\frac{\kappa^2}{4}\frac{\sin^2\phi}{\cosh^2 v}+\frac{\kappa m}{2}\,\cos\phi$$

and all terms are globally defined if we take into account the relation

$$P_v=\frac{\eta_3\,M_1+\eta_1\,M_3}{\sqrt{1+\eta_2^2}}.$$

As a last step we need the constraints on the parameters $(\rho,\,\sigma)$ that ensure the non-vanishing of B for $v\in\mathbb{R}$. These follow from an elementary discussion giving

$$B>0\qquad\Longleftrightarrow\qquad\Big(\,|\sigma|\geq 1\ \wedge\ \rho+1>2|\sigma|\,\Big)\ \vee\ \Big(\,|\sigma|<1\ \wedge\ \rho>\sigma^2\,\Big)$$
$$B<0\qquad\Longleftrightarrow\qquad \rho+1<-2|\sigma|$$

ending up the proof. $\Box$

5 Lower order integrals

An important question is whether all the integrable systems considered in the previous sections admit conserved quantities of degree strictly less than four with respect to the momentum grading. We will write the hamiltonian

$$2H=a^2(x)\,P_x^2+b^2(x)\,P_\phi^2+f(x)\,\cos\phi+g(x)\tag{101}$$

where none of the functions $(a,\,b,\,f)$ can be identically vanishing and were determined in the previous sections. Let us prove:

Theorem 11 *No first degree integral is possible for the hamiltonian (101).*

Proof: Let us write the extra conserved linear integral as

$$R = a(x)\,A(x,\phi)\,P_x + B(x,\phi)\,P_\phi.$$

The constraint $\{H,R\}=0$ involves terms of degree 2 and 0 in the momenta giving

$$\partial_x A = 0 \qquad \partial_x\,B = -\frac{b^2}{a}\,\partial_\phi A \qquad \partial_\phi B = a\,\frac{b'}{b}\,A \qquad B = \frac{a}{f}\,\frac{A}{\sin\phi}(g' + f'\,\cos\phi).$$

Inserting the last relation in the second one we get

$$\left(a\,\frac{f'}{f}\right)'\cos\phi + \left(a\,\frac{g'}{f}\right)' = -\frac{b^2}{a}\,\sin\phi\,\frac{\dot{A}(\phi)}{A(\phi)} \qquad\qquad \dot{A}(\phi) = D_\phi\,A(\phi). \tag{102}$$

We have supposed that $A(\phi)\not\equiv 0$ since then B hence R_1 would vanish identically. The discussion has to consider two cases since $\dot{A}(\phi)$ may be identically vanishing or not. This leads to the constraints

$$(c_1): \qquad \left(a\,\frac{g'}{f}\right)' = 0 \qquad\qquad (c_2): \qquad \frac{a}{b^2}\left(a\,\frac{f'}{f}\right)' = \lambda \qquad \lambda\in\mathbb{R}. \tag{103}$$

For the models in Theorems 2 and 3 we have

$$a\,\frac{g'}{f} = \beta\left(l + \frac{m}{G'}\right)$$

which is a constant only for the globally defined systems of Theorem 4, for which $m=0$ and $\beta=1$. But for these cases we have checked that (c_2) does not hold. In all the remaining cases for which β is not a constant, the condition (c_1) will not hold. $\square$

Theorem 12 *No second or third degree integral is possible for the hamiltonian (101).*

Proof: Let us write the quadratic extra conserved quadratic integral as

$$R_2 = a^2(x)\,A(x,\phi)\,P_x^2 + a(x)\,B(x,\phi)\,P_x\,P_\phi + C(x,\phi)\,P_\phi^2.$$

Let us first notice that there is no loss of generality if we do not include a linear term for the following reason: when one expands

$$\{H, R_2 + R_1\} = \{H,R_2\} + \{H,R_1\}$$

the first piece has linear and cubic terms which are decoupled from the quadratic and zero degree terms coming from the second piece, which were already shown in Lemma 3 to produce no linear integral. Such a useful observation is also valid also for higher degrees.

The constraint $\{H,R_2\}=0$ gives six equations from which we select the following ones

$$\partial_x A = 0 \qquad \partial_x\,B = -\frac{b^2}{a}\,\partial_\phi A \qquad B = 2\,\frac{a}{f}\,\frac{A}{\sin\phi}(g' + f'\,\cos\phi).$$

Inserting the last relation in the second one, we get an equation discussed as in Lemma 3, and leading to the same constraints given by (103) which never hold simultaneously.

There remains to consider the case of a cubic integral

$$R_3 = a^3(x)\,A(x,\phi)\,P_x^3 + a^2(x)\,B(x,\phi)\,P_x^2\,P_\phi + a(x)\,C(x,\phi)\,P_x\,P_\phi^2 + D(x,\phi)\,P_\phi^3.$$

From $\{H,R_3\}=0$ we select the relations

$$\partial_x A = 0 \qquad\qquad \partial_x\,B = -\frac{b^2}{a}\,\partial_\phi A \qquad\qquad B = 3\,\frac{a}{f}\,\frac{A}{\sin\phi}(g' + f'\,\cos\phi)$$

leading again to the constraints (103) and concluding the proof. $\square$

6 Conclusion

Following the ideas put forward by Selivanova [5], [6], [2] a large class of explicit integrable systems with cubic or quartic first integrals has now been obtained with full control of their global structure. However, let us observe that the *local* structure of a larger class of systems with quartic integrals was obtained by Yehia [9]: they models exhibit a more general potential in the hamiltonian and many parameters, but their global structure remains unknown. The approach followed in all of these references relied on a partial differential equation first derived by [3] and which is appropriate for cubic and quartic integrals but does not seem to generalize to higher degree integrals. However from the pioneering work of Kiyohara [4] we know that integrals of any degree do exist for two-dimensional manifolds. In [8] and in this work a more direct analysis of the differential system leading to integrability revealed to be also successful: it will be interesting to see in the future if, following such a path, one can obtain new explicit examples exhibiting integrals of degree strictly larger than four.

Appendix A: notational conventions

The riemannian metrics dealt with in this article have the generic form

$$g = A^2(v)\,dv^2 + B^2(v)\,d\phi^2 \tag{104}$$

Our definitions of the scalar curvature and of the Ricci tensor are

$$R = -\frac{2}{AB}\left(\frac{B'}{A}\right)' \qquad\qquad Ric_{ij} = \frac{R}{2}\,g_{ij}.$$

Let us first consider $\mathbb{R}^3$ equipped with the coordinates $(\,\zeta_1,\,\zeta_2,\,\zeta_3\,)$. Then the two-dimensional sphere $\zeta_1^2+\zeta_2^2+\zeta_3^2=1$ is embedded according to

$$\zeta_1 = \sin\theta\,\cos\phi \qquad \zeta_2 = \sin\theta\,\sin\phi \qquad \zeta_3 = \cos\theta \qquad \theta \in\ (0,\pi)$$

while the angle ϕ is azimuthal, which means that $\phi \in\ [0,2\pi)$ is extended to $[0,2\pi]$ upon identification of $\phi=0$ and $\phi=2\pi$. The induced metric is

$$g(S^2,\mathrm{can}) = d\zeta_1^2+d\zeta_2^2+d\zeta_3^2 = d\theta^2+\sin^2\theta\,d\phi^2 \qquad\qquad R=2. \tag{105}$$

Let us notice that the points $\theta=0$ and $\theta=\pi$ are just apparent coordinate singularities (the "poles") which must be added to get the full manifold $\mathbb{S}^2$.

We will take as generators of $so(3)$ in $T^*\mathbb{S}^2$

$$L_1 = \sin\phi\,P_\theta + \frac{\cos\phi}{\tan\theta}\,P_\phi \qquad L_2 = -\cos\phi\,P_\theta + \frac{\sin\phi}{\tan\theta}\,P_\phi \qquad L_3 = P_\phi$$

with the Poisson brackets

$$\{L_1,L_2\} = -L_3 \qquad\qquad \{L_2,L_3\} = -L_1 \qquad\qquad \{L_3,L_1\} = -L_2$$

which allow to write the hamiltonian

$$2H = P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta} = L_1^2+L_2^2+L_3^2.$$

Similarly $\mathbb{H}^2$ which is defined, taking $(\eta_1,\,\eta_2,\,\eta_3)$ for the coordinates in $\mathbb{R}^3$, by

$$\eta_1^2+\eta_2^2-\eta_3^2=-1 \qquad (\eta_1,\eta_2)\in\mathbb{R}^2 \qquad \eta_3\geq 1$$

will be embedded according to

$$\eta_1 = \sinh\chi\,\cos\phi \qquad \eta_2 = \sinh\chi\,\sin\phi \qquad \eta_3 = \cosh\chi \qquad \chi \in\ (0, +\infty)$$

where ϕ is again azimuthal. The induced metric is

$$g(H^2, \mathrm{can}) = d\eta_1^2 + d\eta_2^2 - d\eta_3^2 = d\chi^2 + \sinh^2\chi\,d\phi^2 \qquad\qquad R = -2. \qquad (106)$$

This time the point $\chi = 0$ is again an apparent coordinate singularity, which must be added to get the manifold $\mathbb{H}^2$, while at infinity the metric takes the characteristic form

$$g \sim du^2 + e^u\,d\phi^2 \qquad\qquad u = 2\chi \to +\infty$$

We will take as generators of $so(2,1)$ in $T^*\mathbb{H}^2$

$$M_1 = \sin\phi\,P_\chi + \frac{\cos\phi}{\tanh\chi}\,P_\phi \qquad M_2 = -\cos\phi\,P_\chi + \frac{\sin\phi}{\tanh\chi}\,P_\phi \qquad M_3 = P_\phi$$

with the Poisson brackets

$$\{M_1, M_2\} = M_3 \qquad\qquad \{M_2, M_3\} = -M_1 \qquad\qquad \{M_3, M_1\} = -M_2$$

which give for hamiltonian

$$2H = P_\chi^2 + \frac{P_\phi^2}{\sinh^2\chi} = M_1^2 + M_2^2 - M_3^2.$$

As is well known, the embedding of $\mathbb{H}^2$ in $\mathbb{R}^3$ is not unique and as we will experience in Theorem 10 the following embedding

$$\eta_1 = \cosh v\,\sinh\phi \qquad \eta_2 = \sinh v \qquad \eta_3 = \cosh v\,\cosh\phi \qquad (v,\,\phi) \in\,\mathbb{R}^2$$

is useful. The induced metric is

$$g(H^2, \mathrm{can}) = d\eta_1^2 + d\eta_2^2 - d\eta_3^2 = dv^2 + \cosh^2 v\,d\phi^2 \qquad\qquad R = -2 \qquad (107)$$

and the generators of $so(2,1)$ in $T^*\mathbb{H}^2$ become

$$M_1 = \cosh\phi\,P_v - \tanh v\,\sinh\phi\,P_\phi \qquad M_2 = P_\phi \qquad M_3 = -\sinh\phi\,P_v + \tanh v\,\cosh\phi\,P_\phi$$

giving for hamiltonian

$$2H = P_v^2 + \frac{P_\phi^2}{\cosh^2 v} = M_1^2 + M_2^2 - M_3^2.$$

References

[1] D. N. Goryachev, *Varshavskie Universitet'skie Izvestiya*, **11** (1916) 3-15.

[2] K. P. Hadeler and E. N. Selivanova, *Regular Chaotic Dyn.*, **3** (1999) 45-52.

[3] L. S. Hall, *Physica D*, **8** (1983) 90-116.

[4] K. Kiyohara, *Math. Ann.*, **320** (2001) 487-505.

[5] E. N. Selivanova, *Commun. Math. Phys.*, **207** (1999) 291-310.

[6] E. N. Selivanova, *Ann. Global Anal. Geom.*, **17** (1999) 201-219.

[7] A. V. Tsiganov, *J. Phys. A: Math. Gen.*, **38** (2005) 3547-3553.

[8] G. Valent, *Commun. Math. Phys.*, **299** (2010) 631-649.

[9] H. M. Yehia, *J. Phys. A: Math. Gen.*, **39** (2006) 5807-5824.